Mathematical Induction

Part Two

Today

- ***Finish up Monday's introduction to Induction***
- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$\boldsymbol{P(0)}$ **is true**

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

How Not To Induct

Something's Wrong...

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For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

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What's wrong with this proof?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then your answer.

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Where did we prove the base case?

Therefore, $P(k + 1)$ is true, completing the induction. ■

When writing a proof by induction,

make sure to prove the base case!

Otherwise, your argument is invalid!

Why did this work?

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We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is 2^{k+1}. To see this, notice that

2^0

You can prove ***anything*** from a faulty assumption. This is called the **principle of explosion**. To see why, read "Animal, Vegetable, or Minister" for a silly example.

Therefore, $P(k + 1)$ is true, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will <u>not</u> explicitly define the $P(n)$.
- But the proof still needs to show
  - that you the author clearly understand what P(n) you're working from :-), and that
  - $P(0)$ is true; and that
  - whenever $P(k)$ is true, $P(k+1)$ is true
- The next two slides show a before & after, where the "after" is how we expect you to write your proofs.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum

Before
(explicit P(n))

powers of two is $2^n - 1$." We will prove, b
$P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

After
(implicit but still clear P(n))

Theorem: The sum of the first n powers of

***Proof*:** By induction.

For our base case, we'll prove the theorem is true when $n = 0$. The sum of the first zero powers of two is zero, and $2^0 - 1 = 0$, so the theorem is true in this case.

For the inductive step, assume the theorem holds when $n = k$ for some arbitrary $k \in \mathbb{N}$. Then

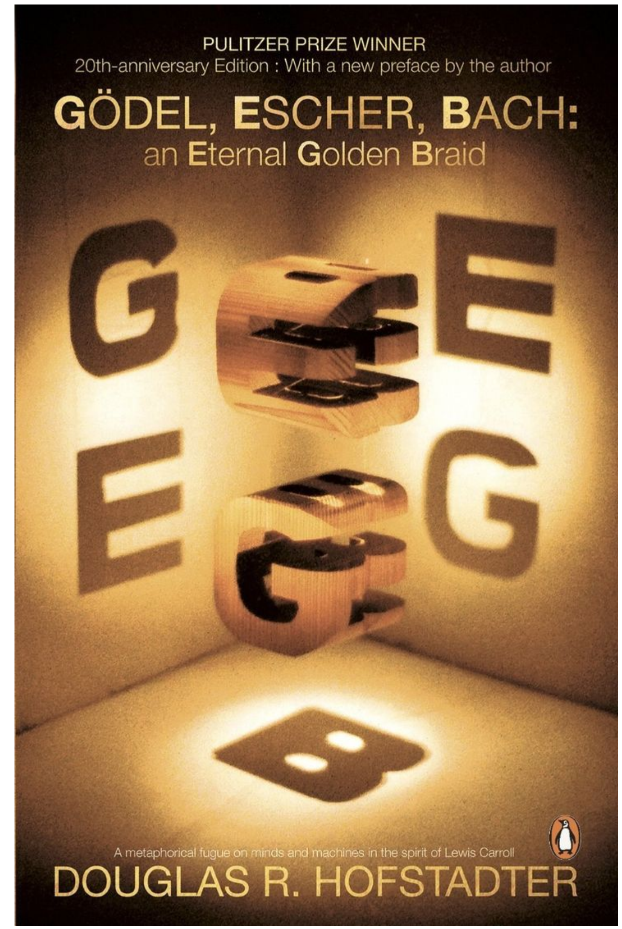
$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

So the theorem is true when $n = k+1$, completing the induction. ■

The `MU` Puzzle

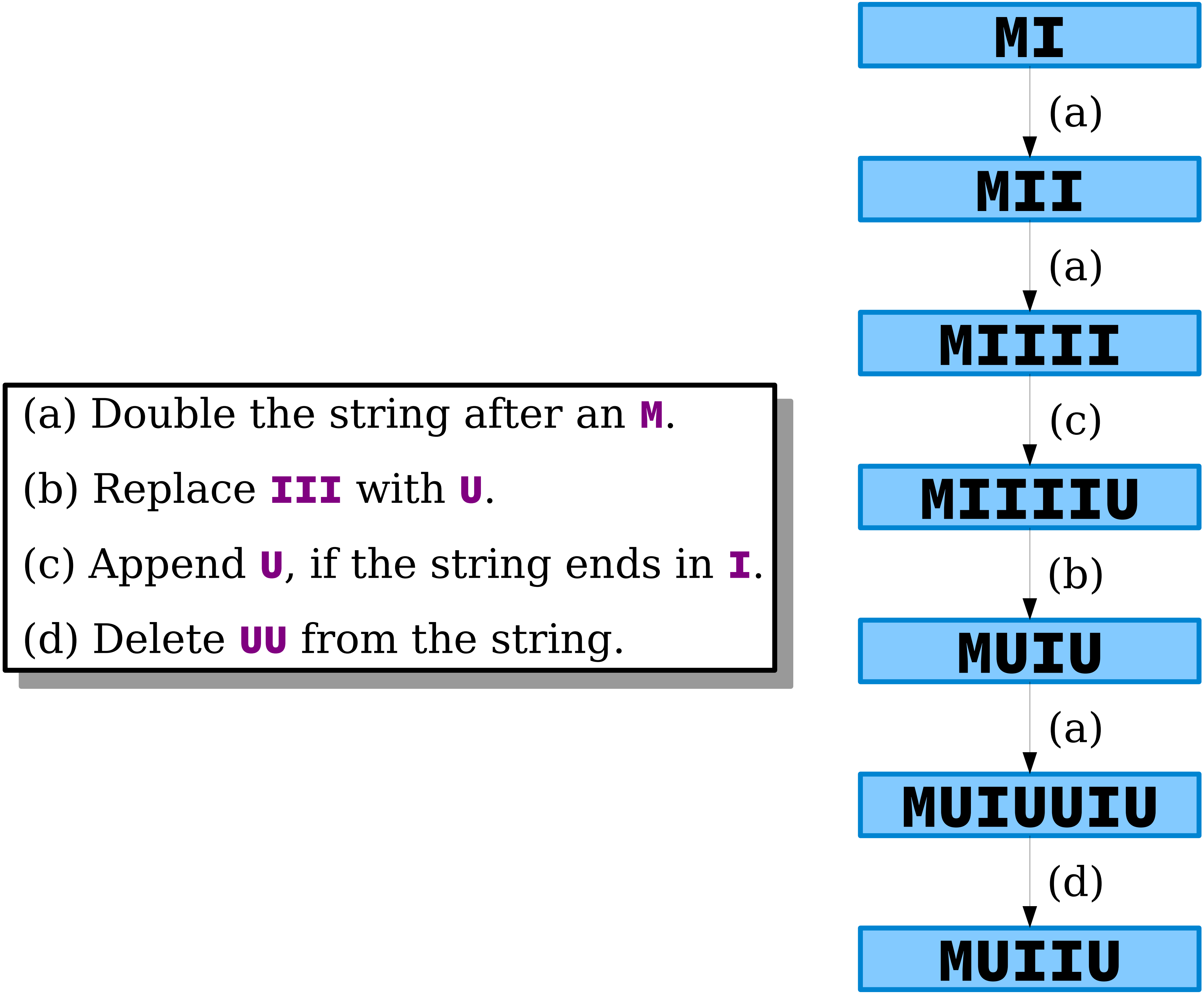
Gödel, Escher Bach: An Eternal Golden Braid

- Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.
- It's a great read after you've finished CS103 – you'll see so many of the ideas we'll cover presented in a totally different way!



The **MU** Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **M**: for example, **MIIU** becomes **MIIUIIU**, or **MI** becomes **MII**.
  - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**.
  - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**.
  - Remove any **UU**: **MUUU** becomes **MU**.
- ***Question***: How do you transform **MI** to **MU**?


(a) Double the string after an M.
(b) Replace III with U.
(c) Append U, if the string ends in I.
(d) Delete UU from the string.
MI
(a)
MII
(a)
MIIII
(c)
MIIIIU
(b)
MUIU
(a)
MUIUUIU
(d)
MUIIU

Try It!

Starting with **`MI`**, apply these operations to make **`MU`**:

(a) Double the string after an **`M`**.

(b) Replace **`III`** with **`U`**.

(c) Append **`U`**, if the string ends in **`I`**.

(d) Delete **`UU`** from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of `I`'s is *not* a multiple of three.
- To make `MU`, the number of `I`'s must end up as a multiple of three.
- Can we *ever* make the number of `I`'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

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Proof: By contrapositive; we'll prove that if $n - 3$ is a multiple of three, then n is also a multiple of three. Because $n - 3$ is a multiple of three, we can write $n - 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

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For our inductive step, suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$. We'll prove $P(k+1)$ is also true. Consider any sequence of $k+1$ moves. Let r be the number of **`I`**'s in the string after the kth move. By our inductive hypothesis (that is, $P(k)$), we know that r is not a multiple of three. Now, consider the four possible choices for the $k+1^{\text{st}}$ move:

Case 1: Double the string after the **`M`**.

Case 2: Replace **`III`** with **`U`**.

Case 3: Either append **`U`** or delete **`UU`**.

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Lemma: No matter which moves are made, the number of **`I`**'s in the string never becomes multiple of three.

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Theorem: The **`MU`** puzzle has no solution.

Proof: Assume for the sake of contradiction that the **`MU`** puzzle has a solution and that we can convert **`MI`** to **`MU`**. This would mean that at the very end, the number of **`I`**'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of **`I`**'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the **`MU`** puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Variations on Induction: ***Starting Later***

Induction Starting at 0

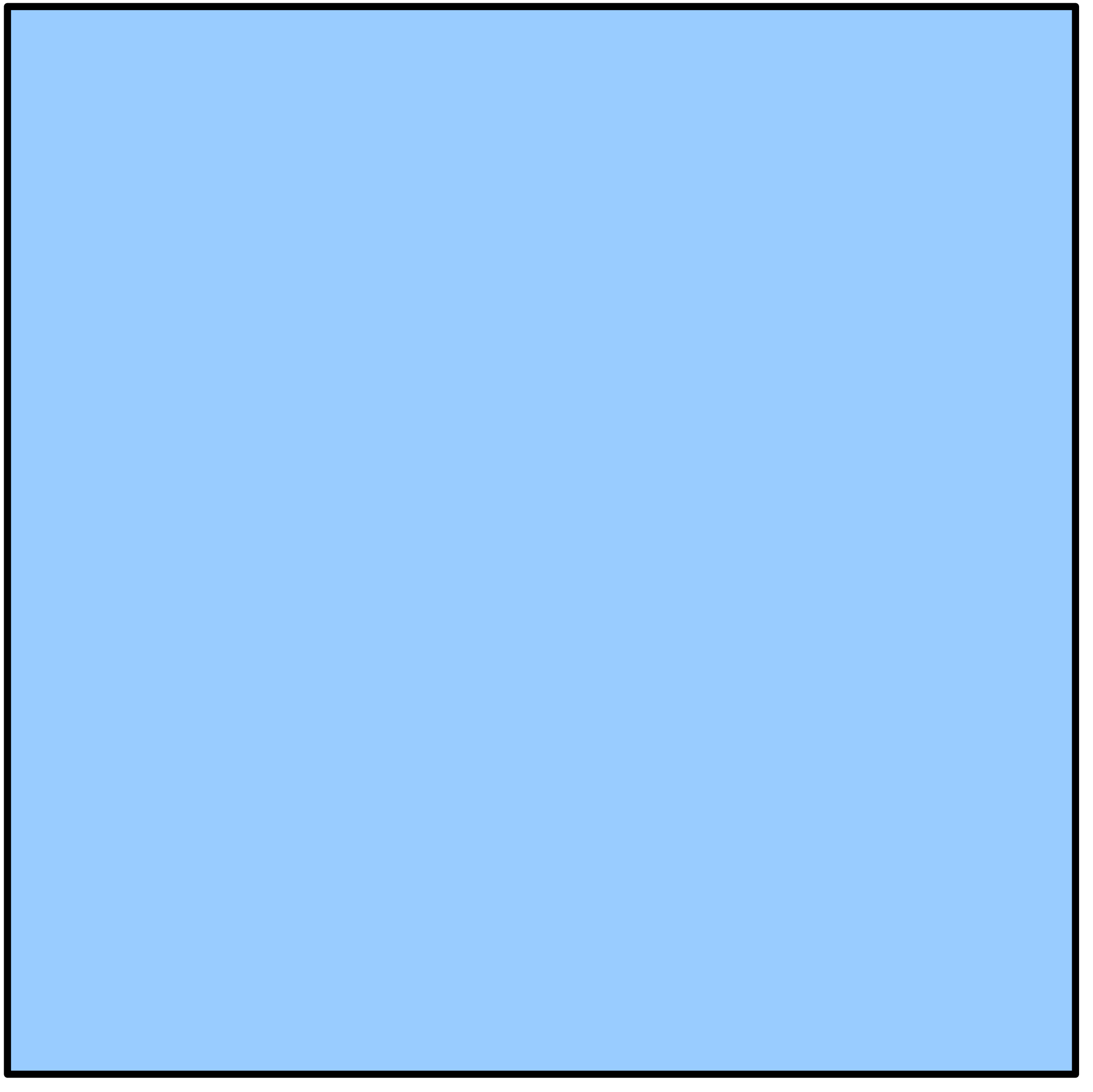
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

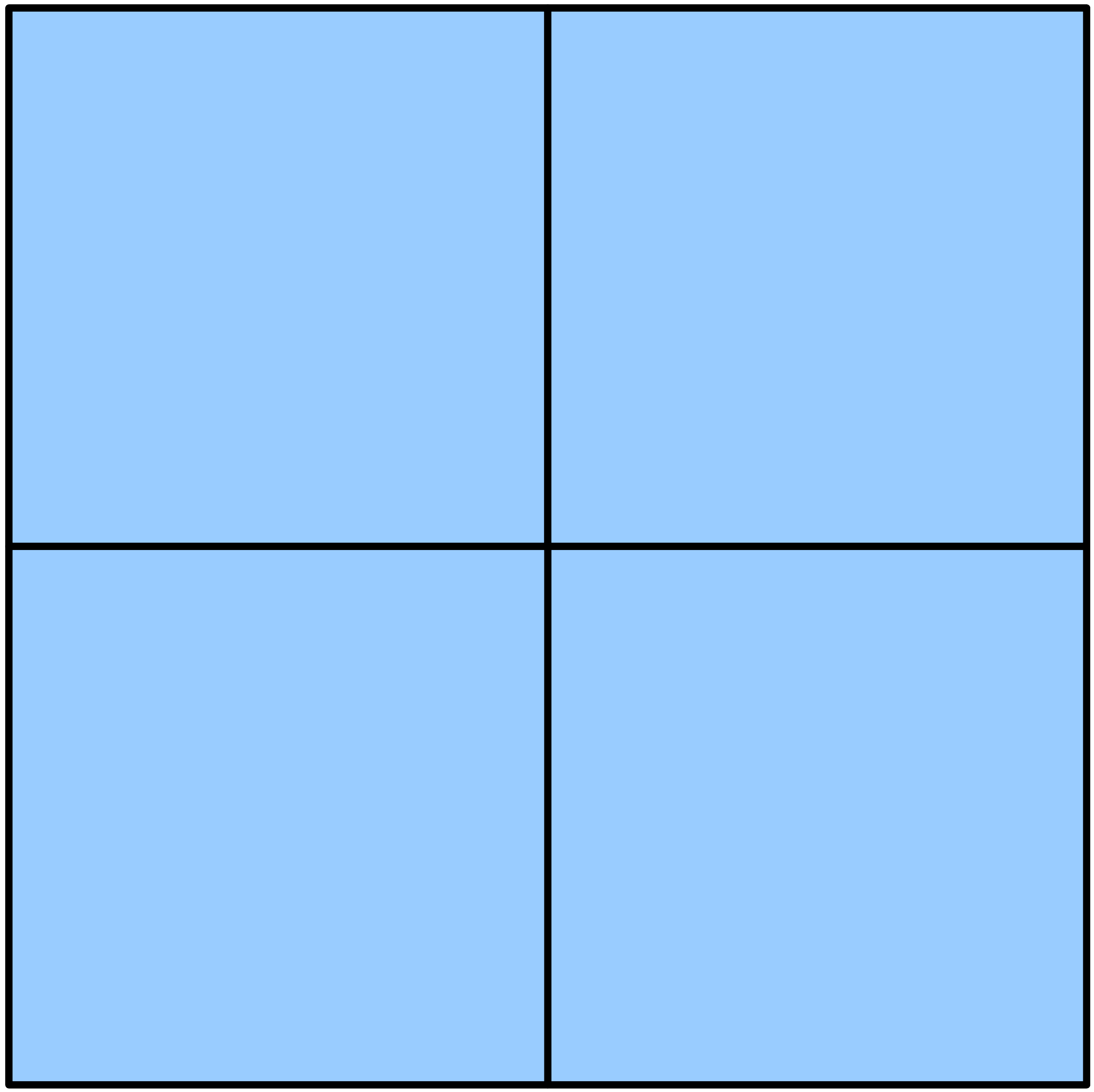
- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

Subdividing a Square



Subdividing a Square



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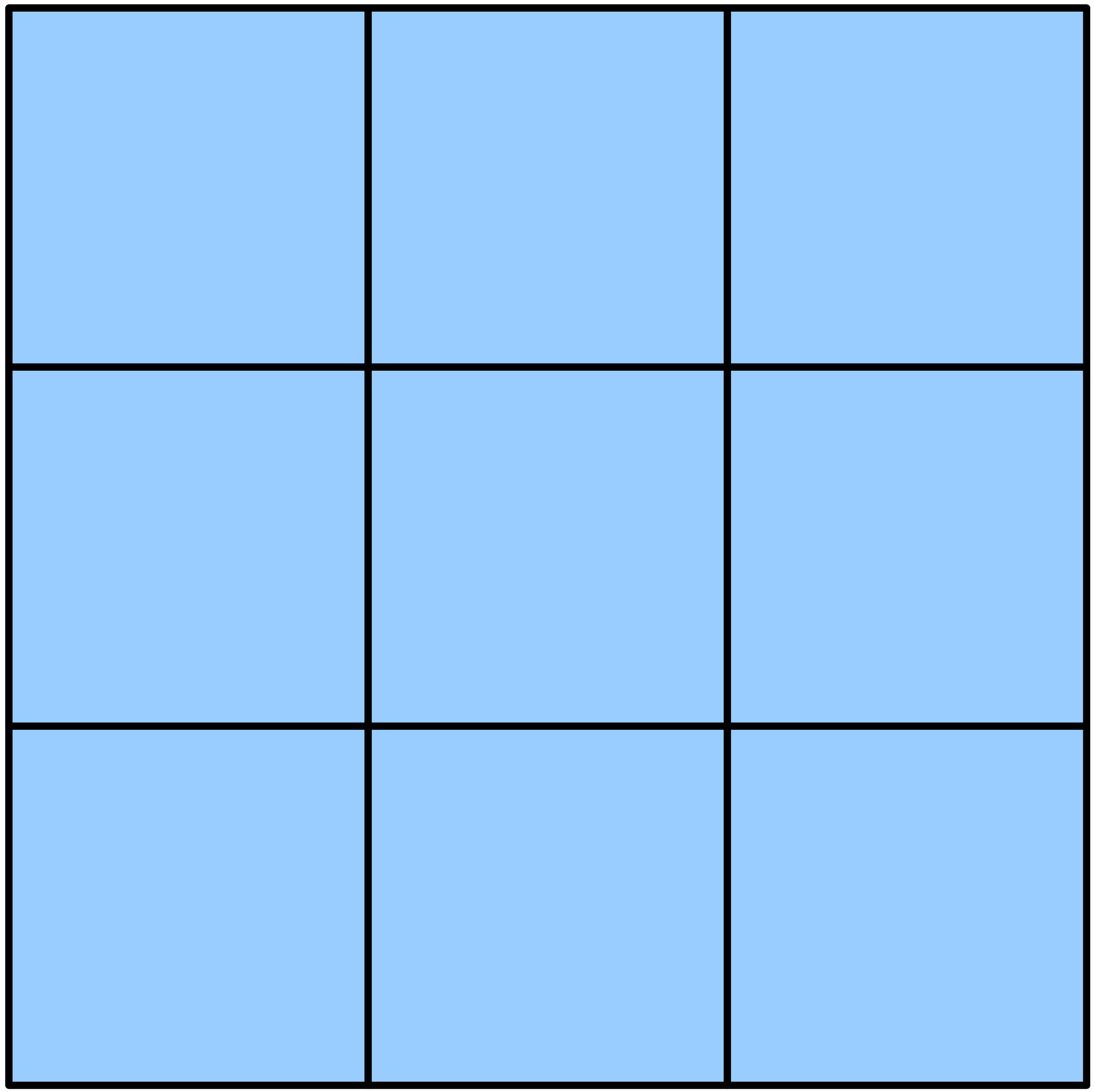

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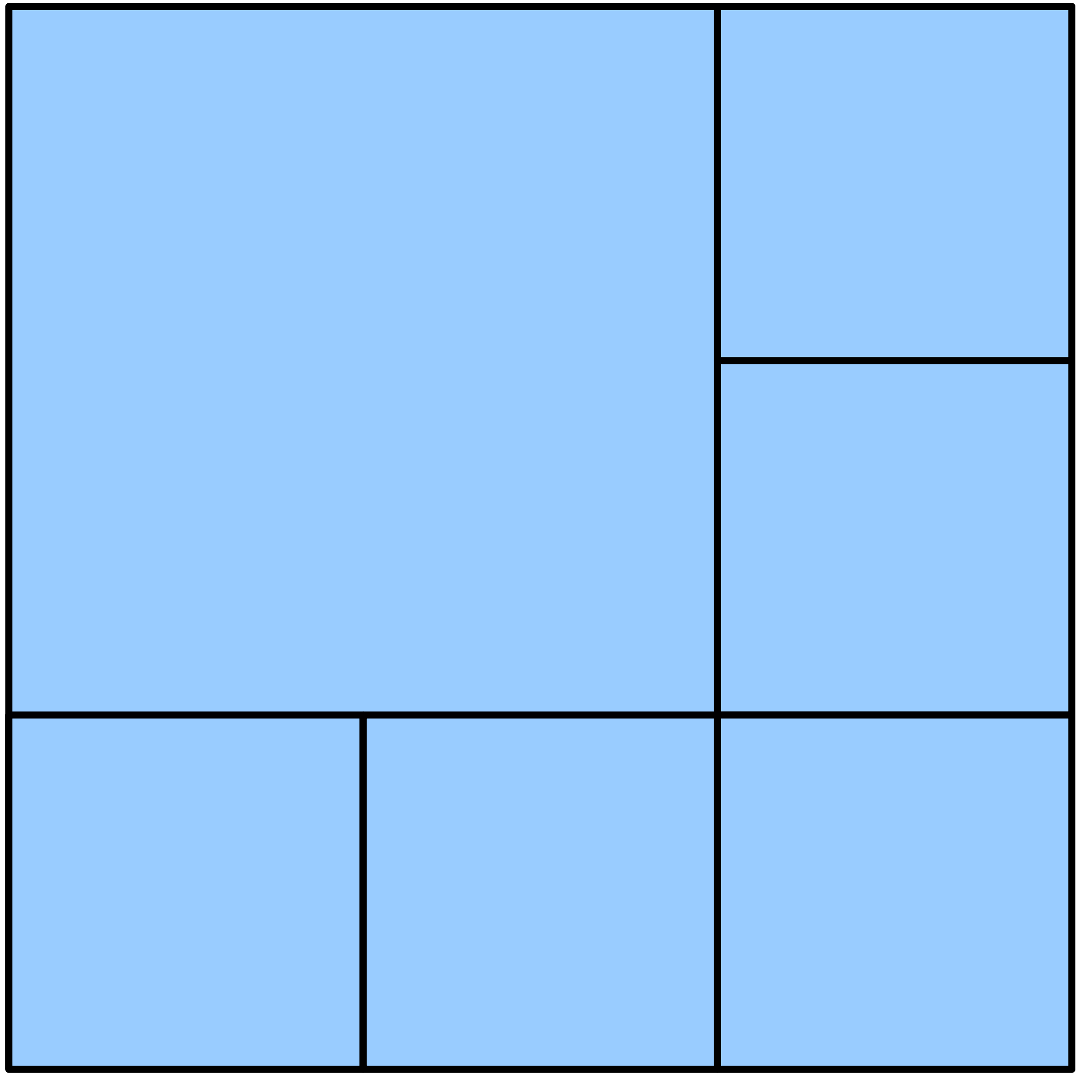

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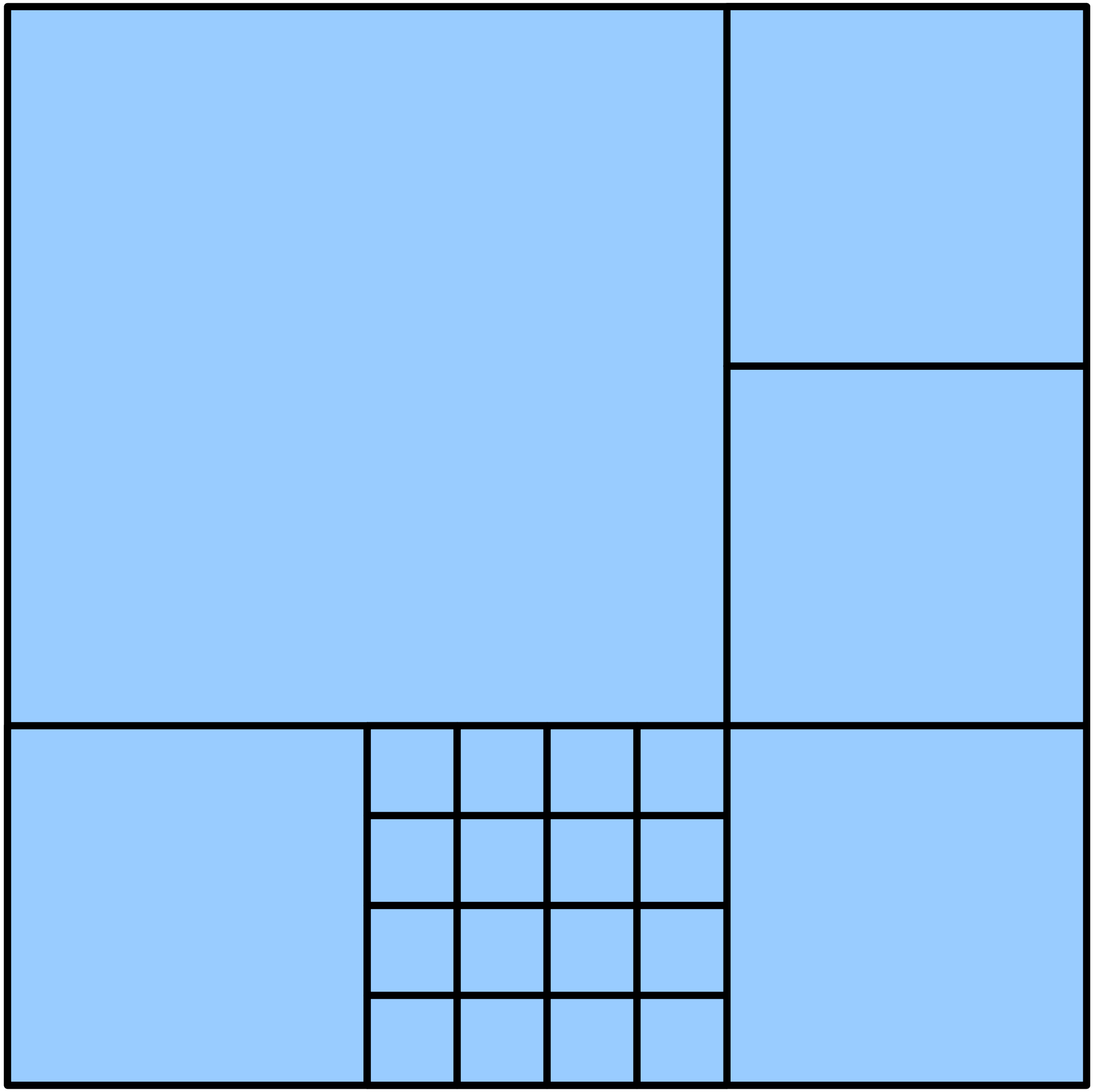


For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

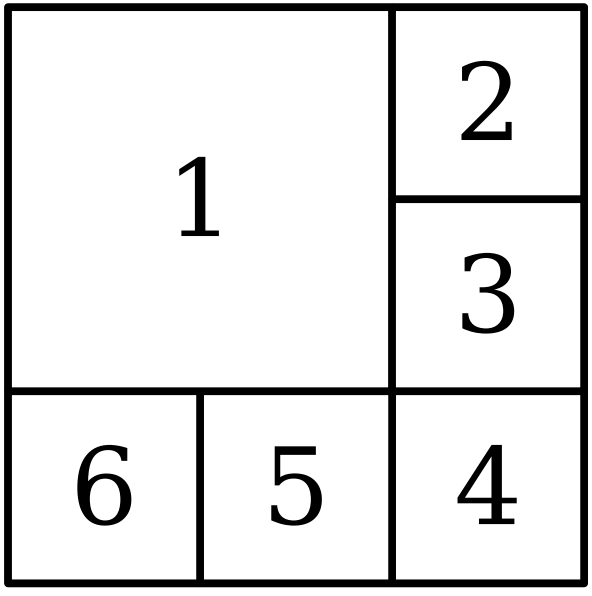
1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2
4	3

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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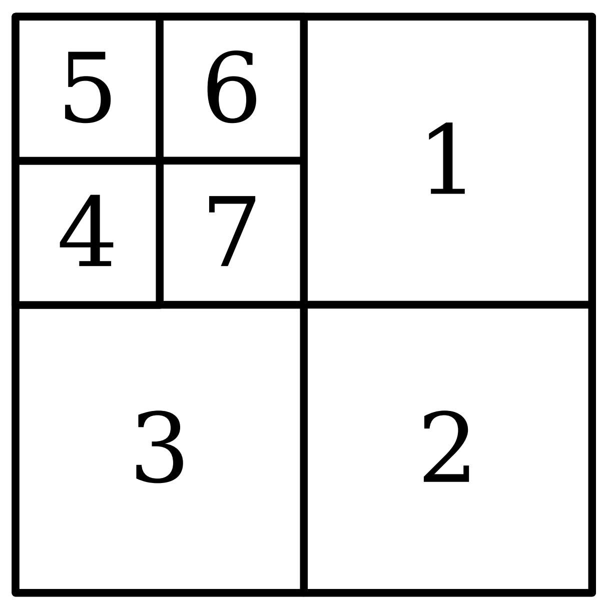

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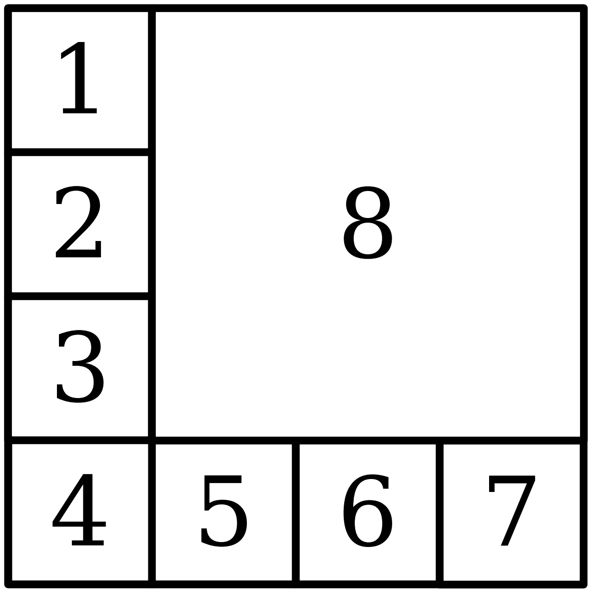
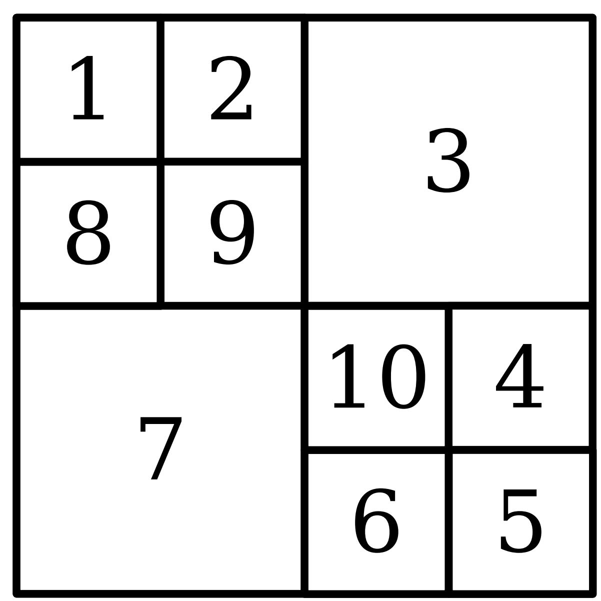


1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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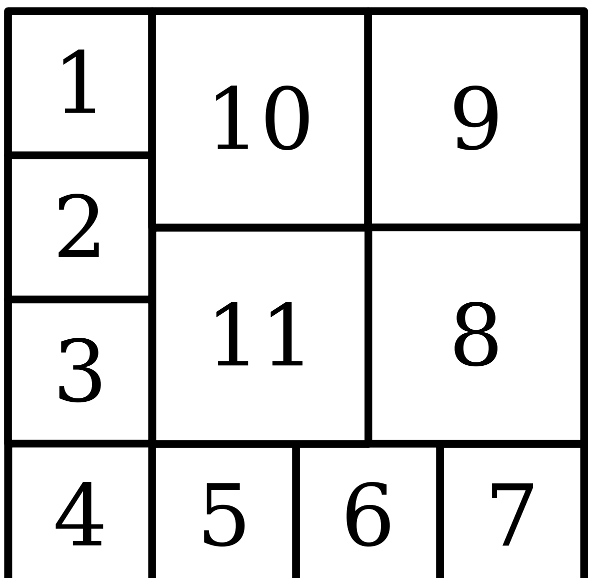

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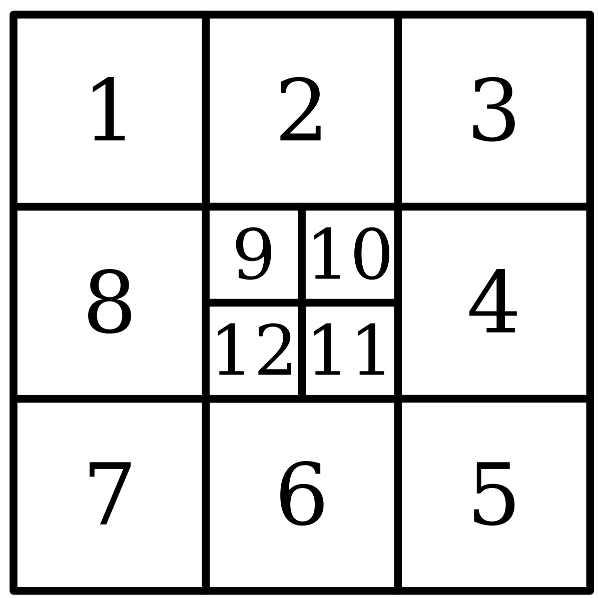

The Key Insight

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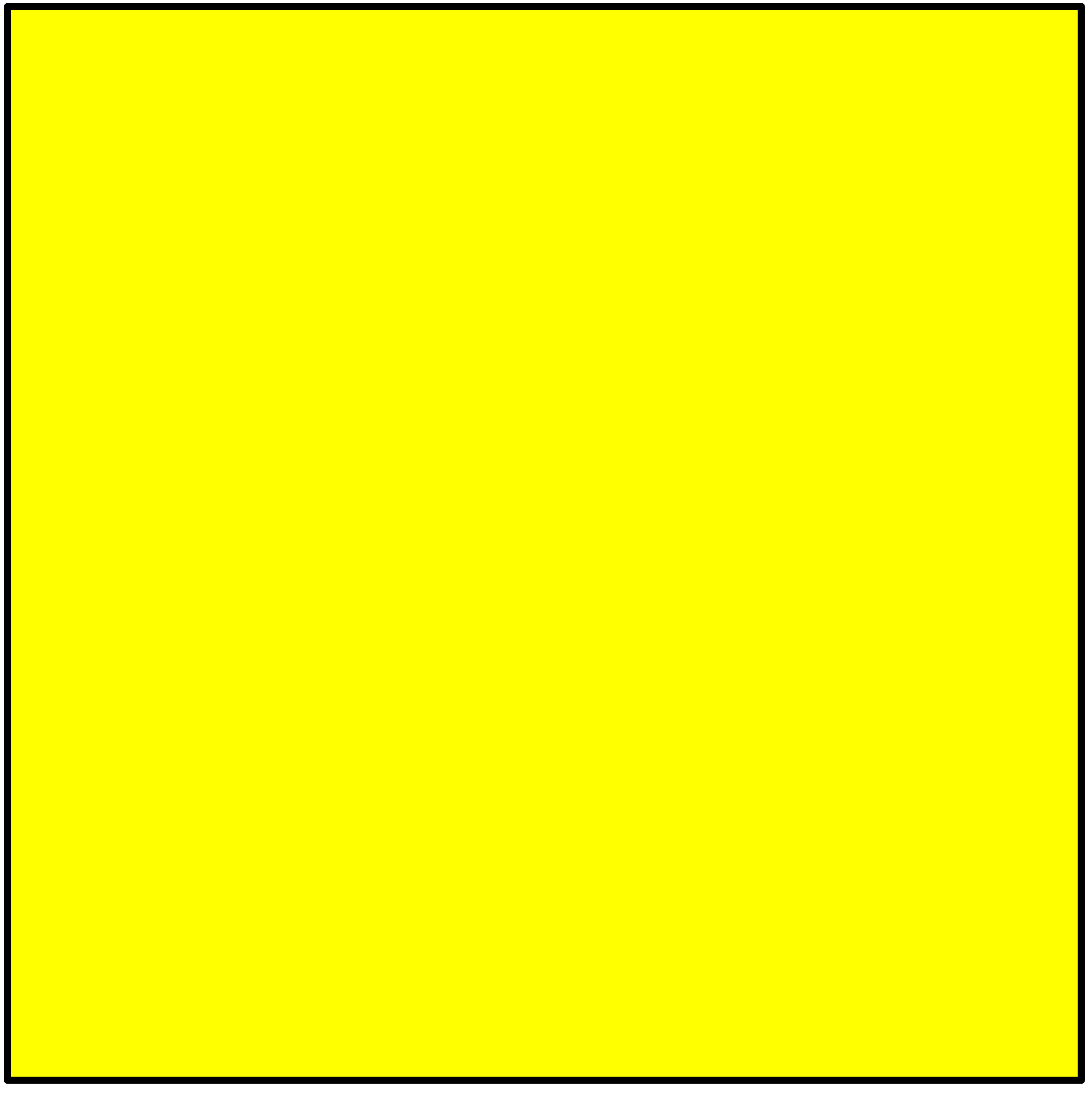

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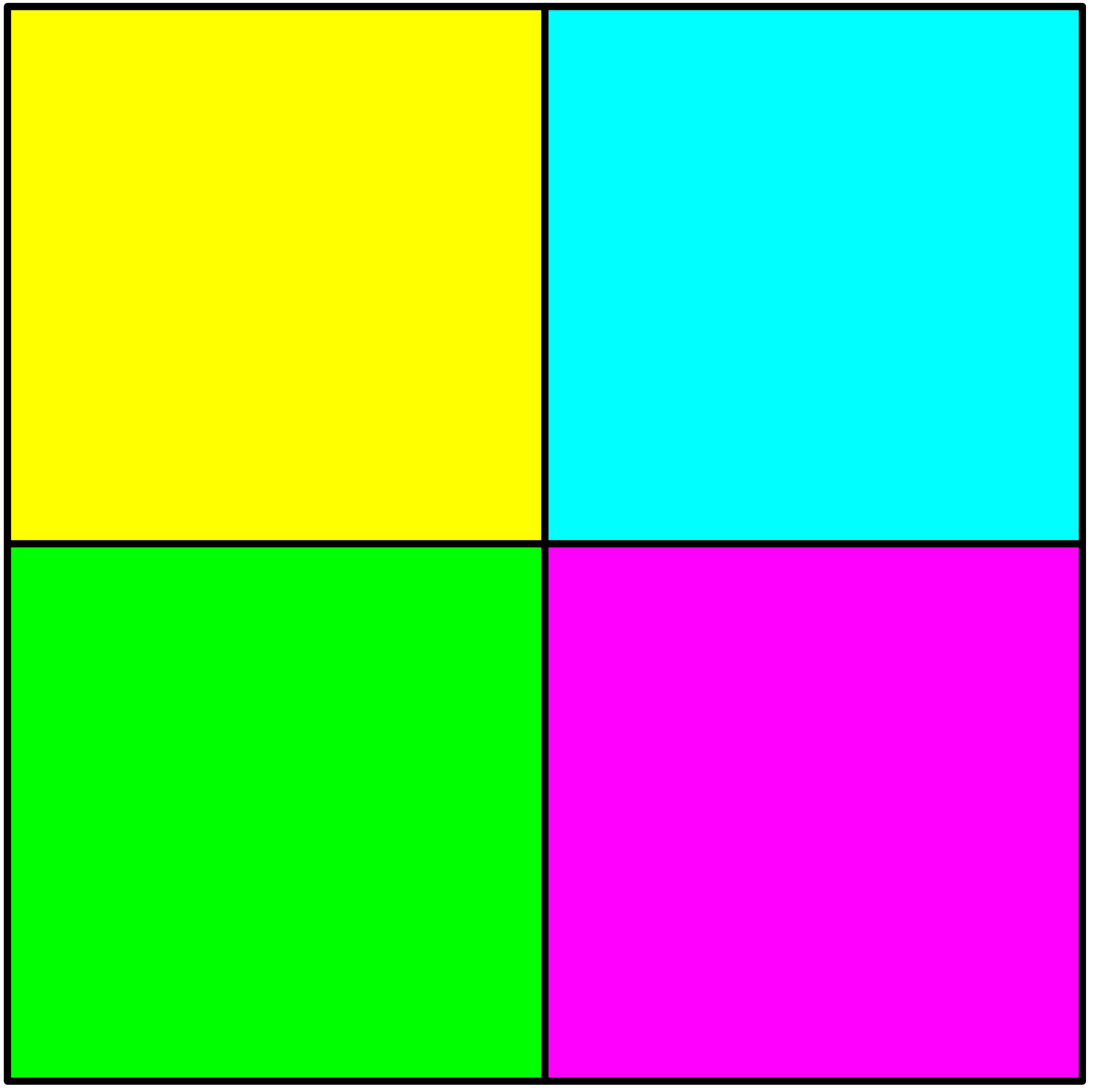

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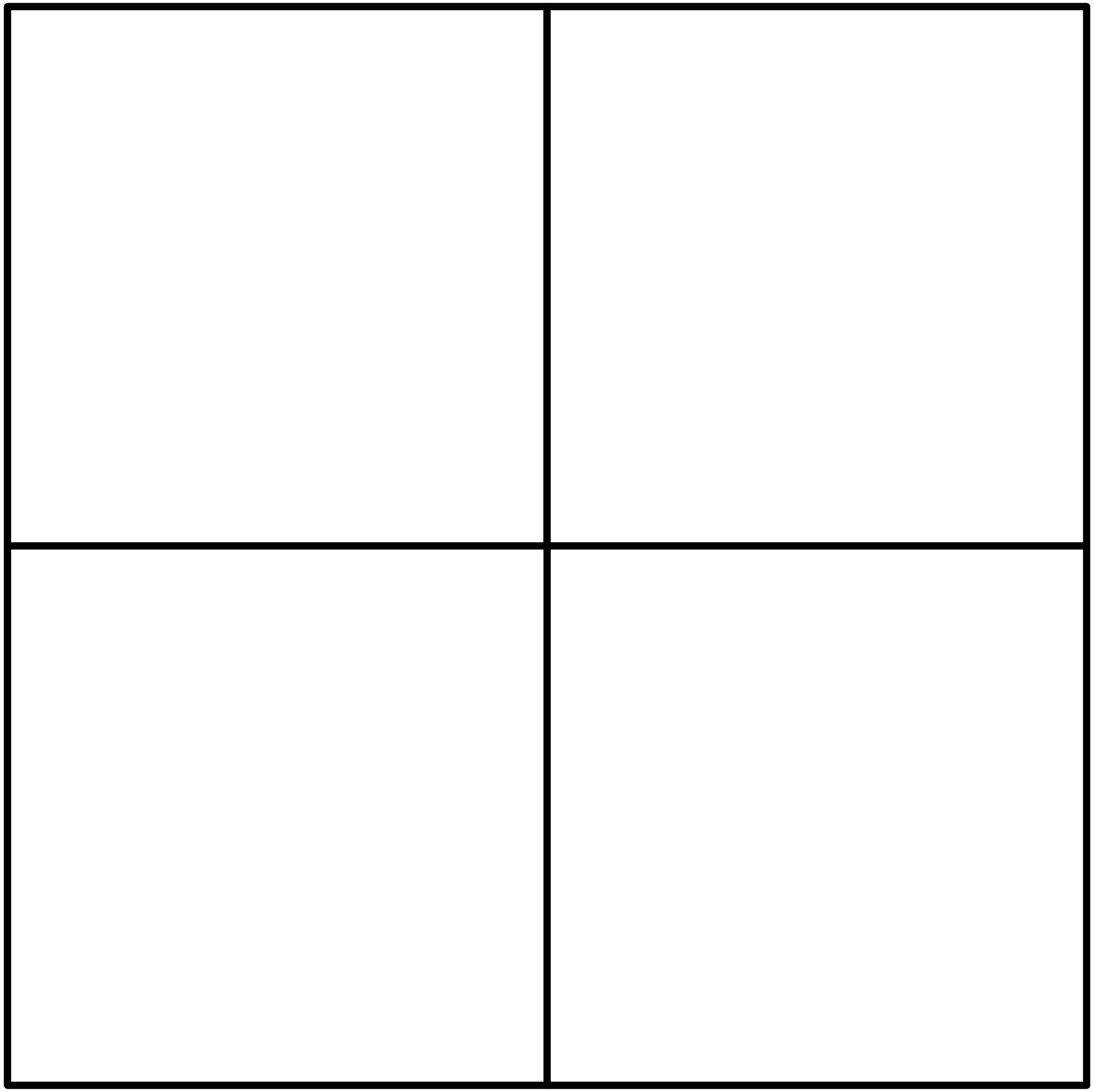

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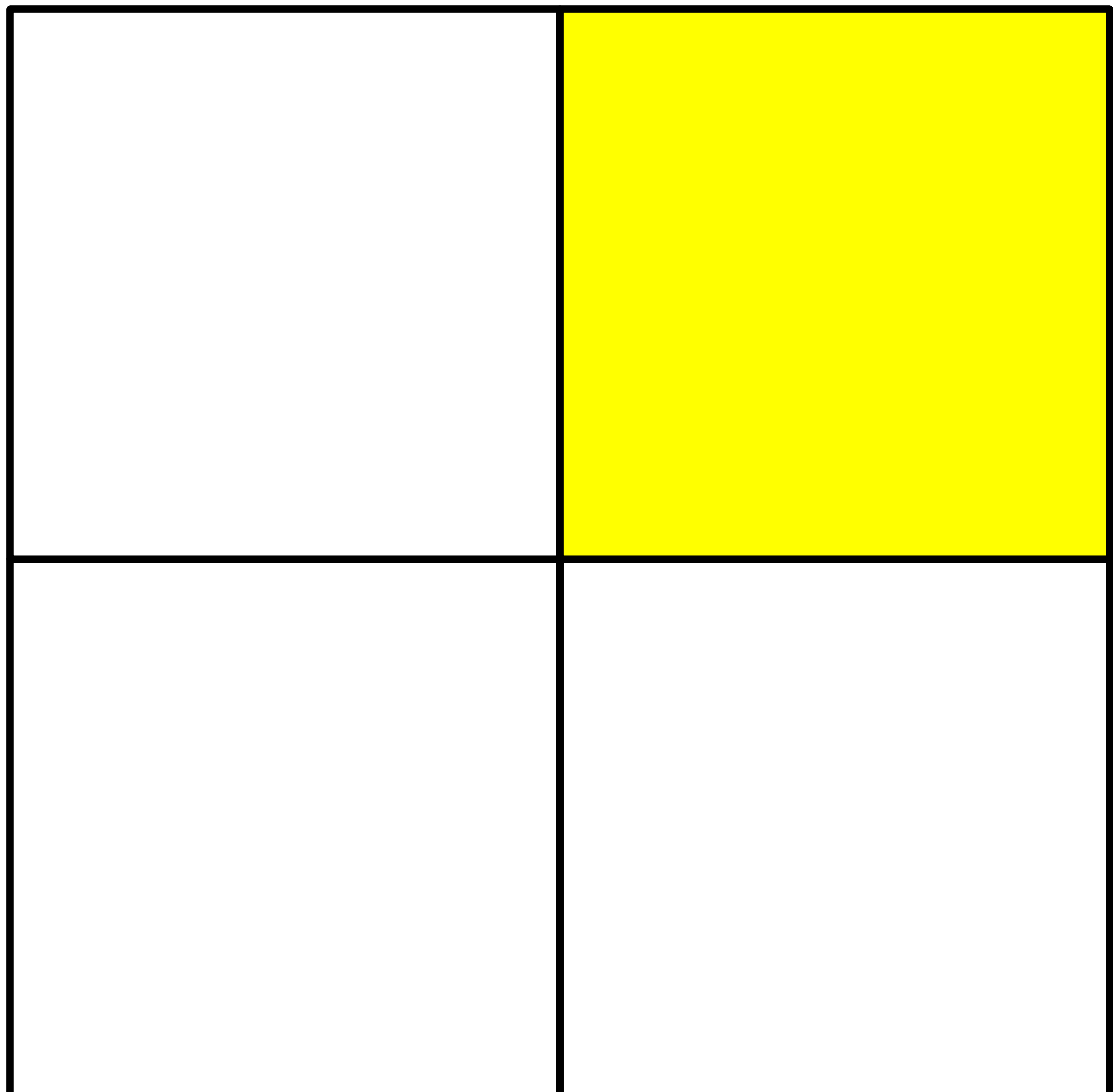

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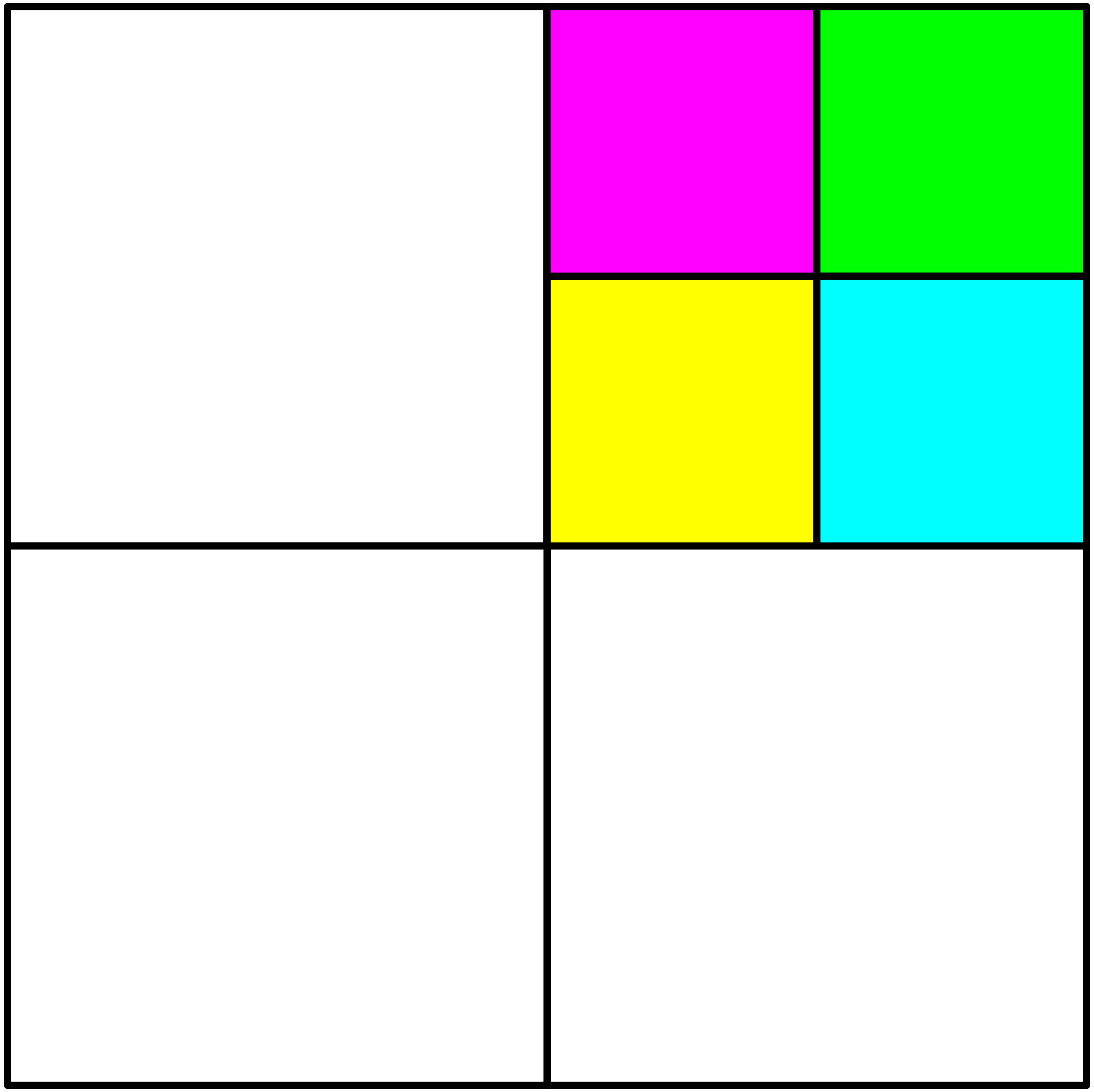

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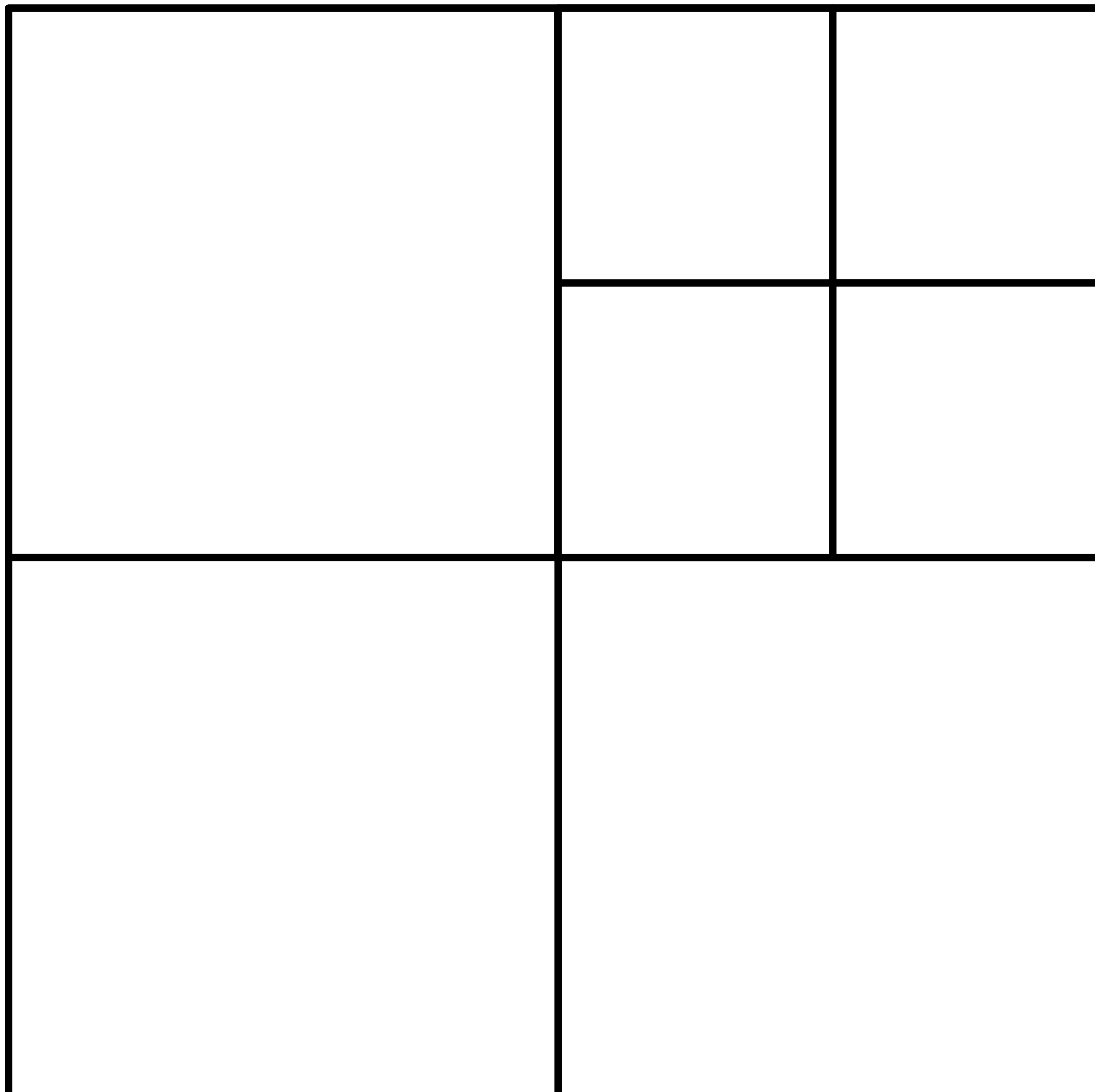

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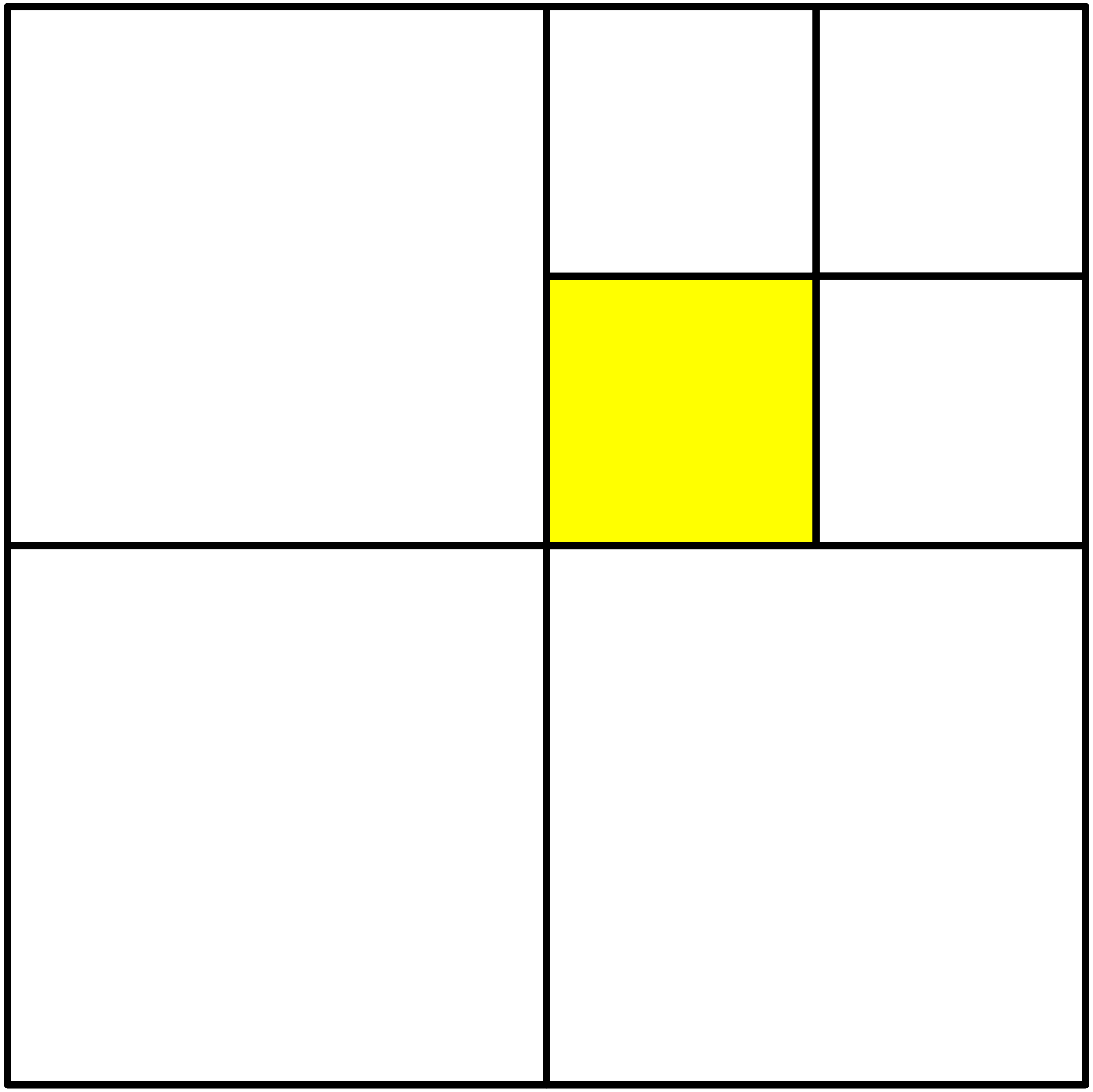

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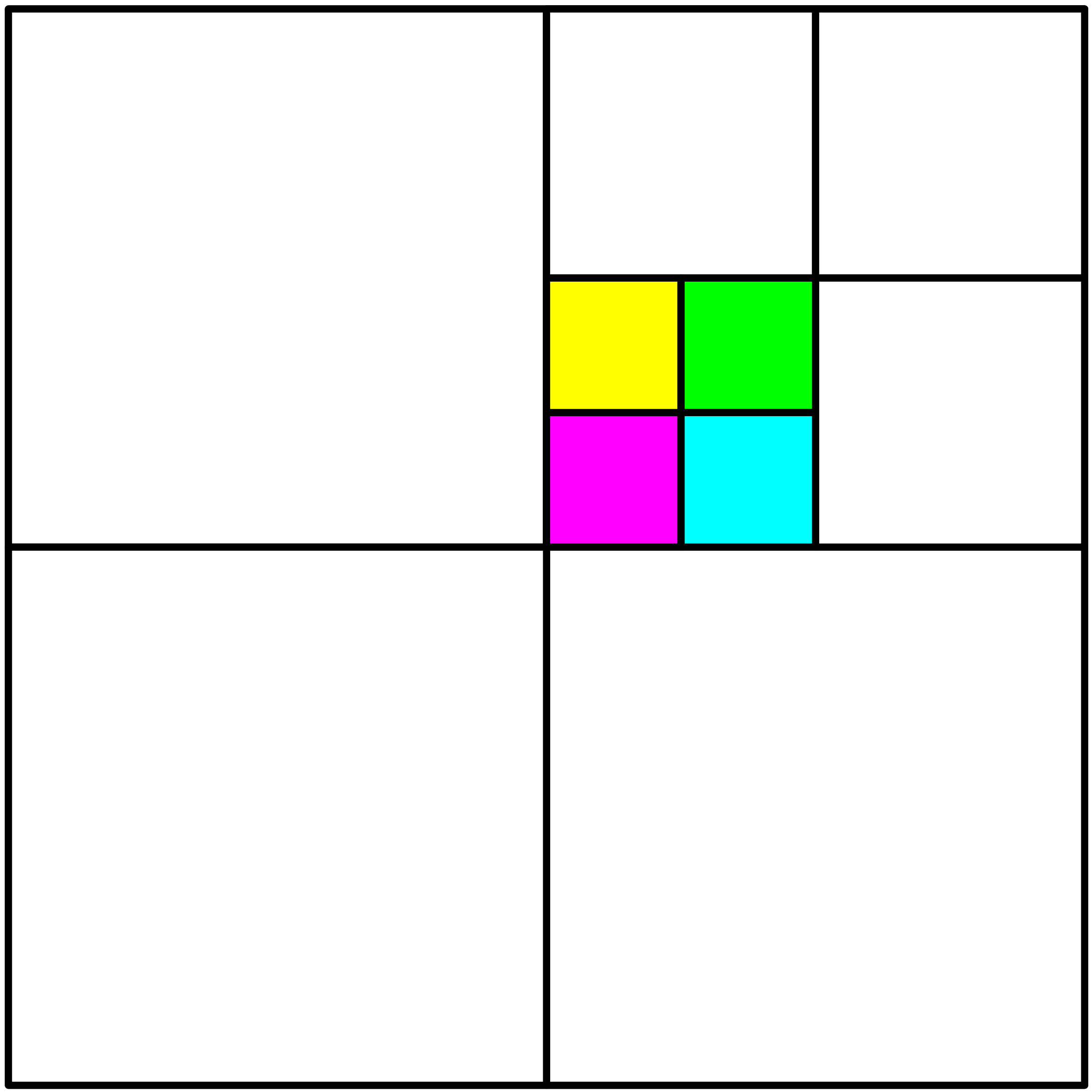

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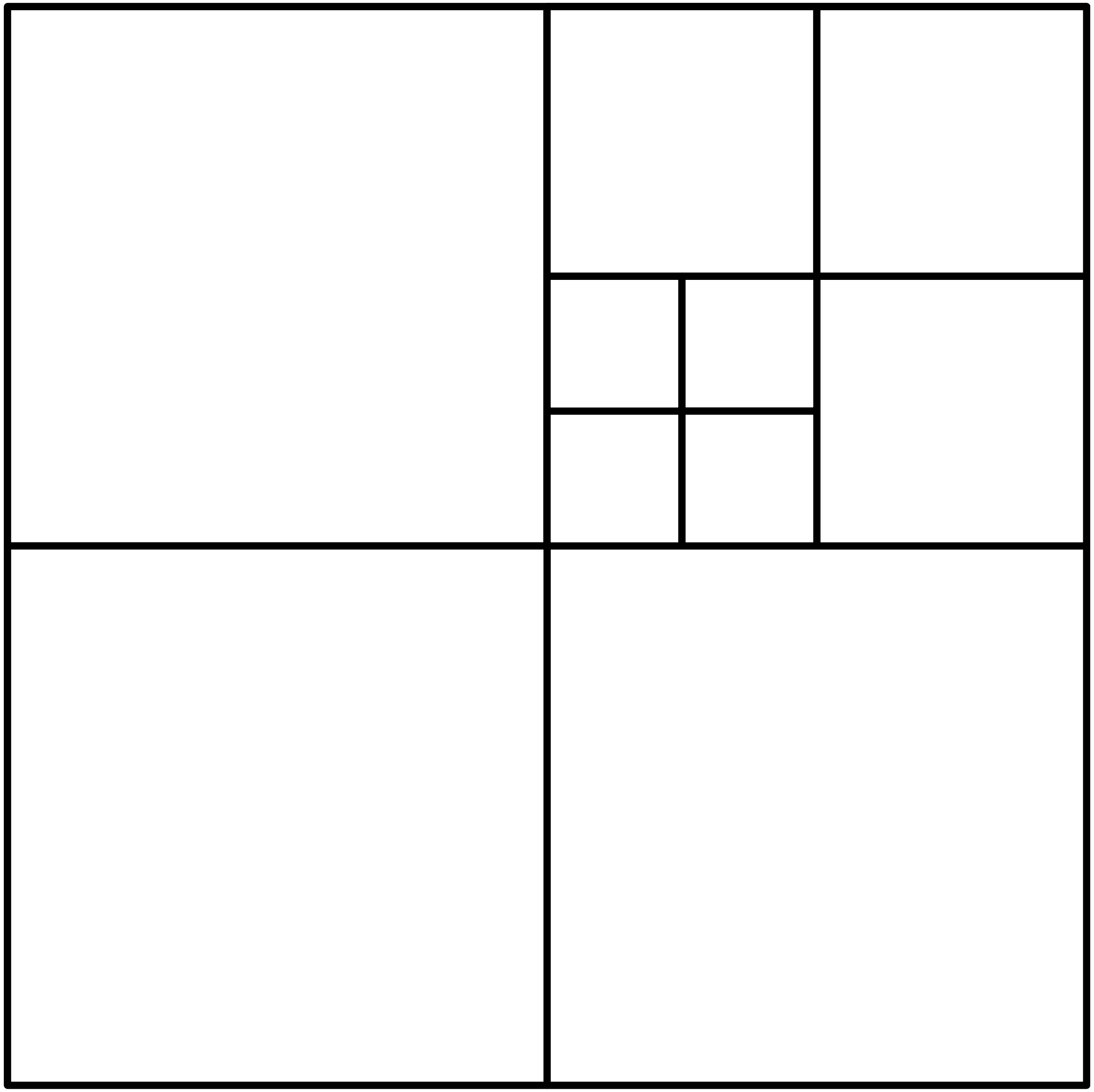

The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, it is possible to subdivide a square into n smaller squares.

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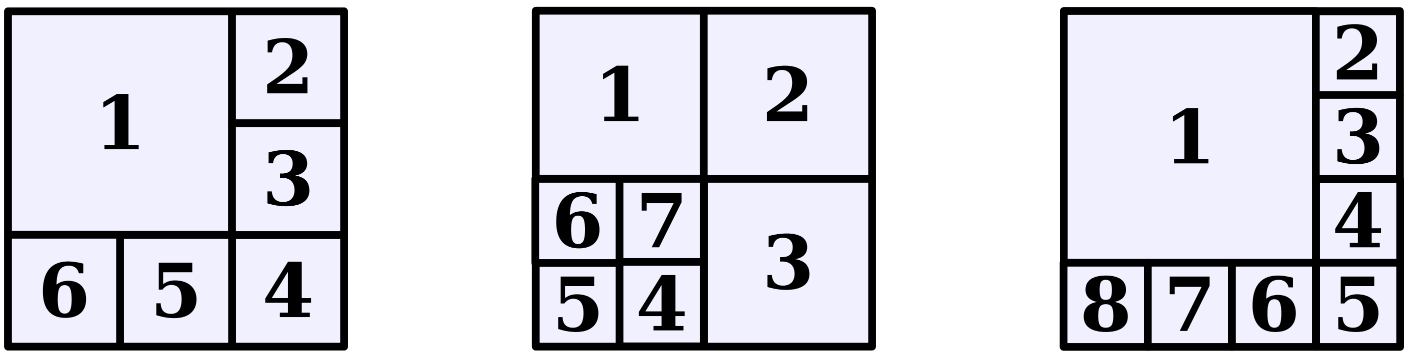

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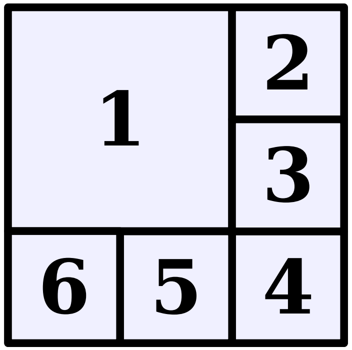

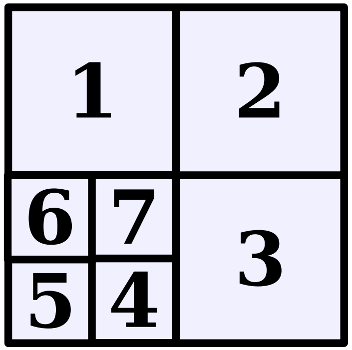

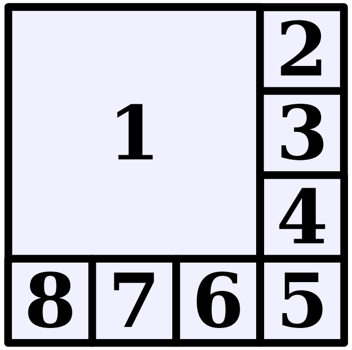

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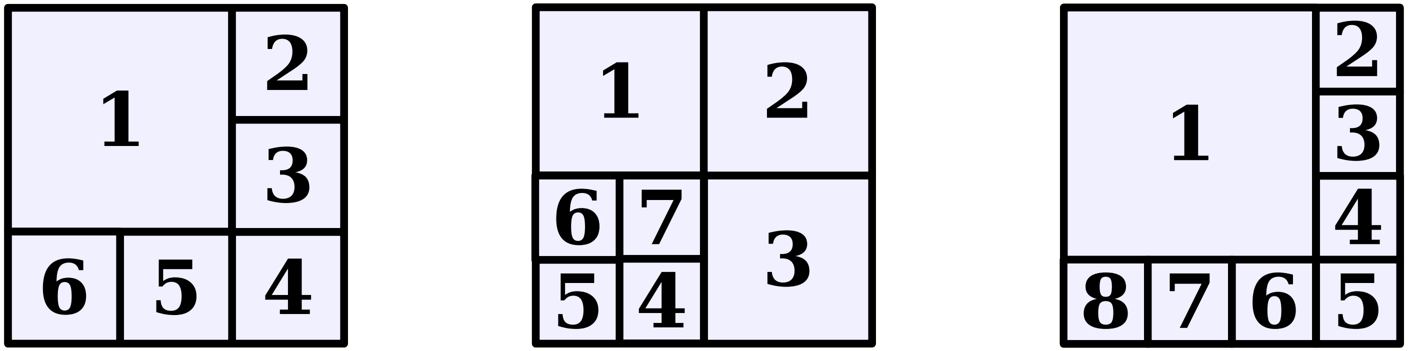

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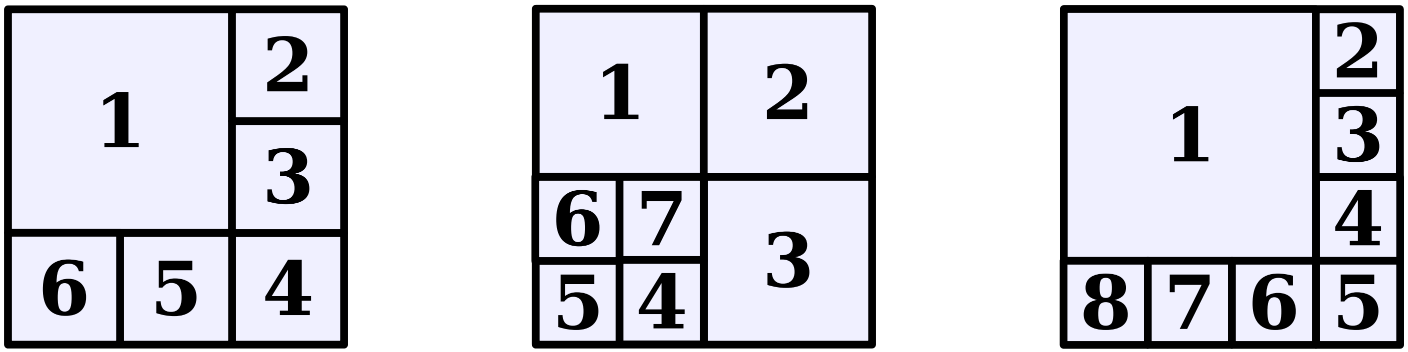

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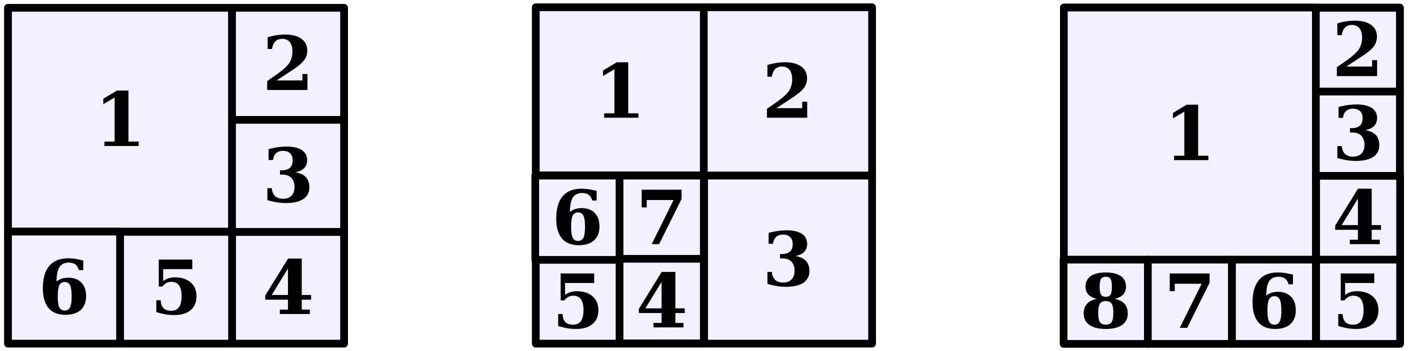

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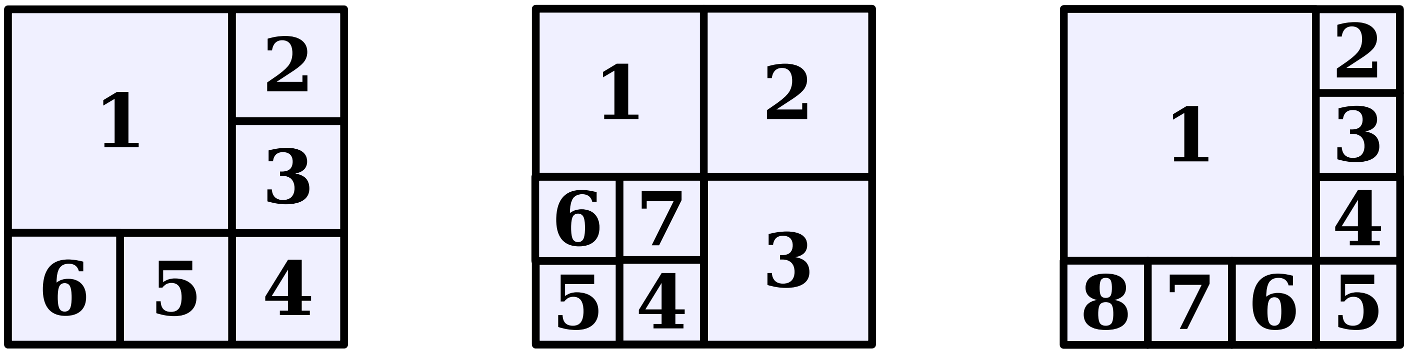

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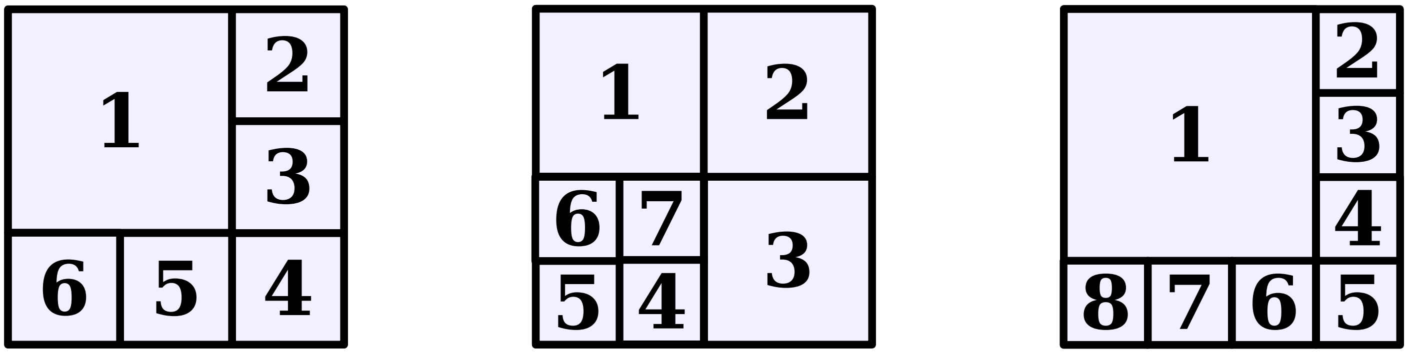

For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that a square can be subdivided into k squares. We prove $P(k+3)$, that a square can be subdivided into $k+3$ squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into k squares. Then, choose any of the squares and split it into four equal squares. This removes one of the k squares and adds four more, so there will be a net total of $k+3$ squares. Thus $P(k+3)$ holds, completing the induction.

Theorem: For any $n \geq 6$, it is possible to subdivide a square into n smaller squares.

Proof: Let $P(n)$ be the statement "a square can be subdivided into n smaller squares." We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

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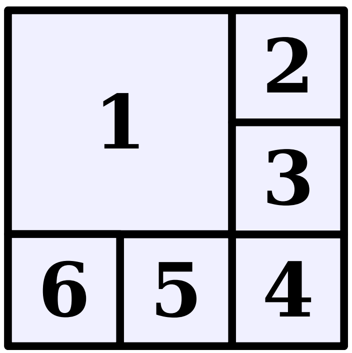

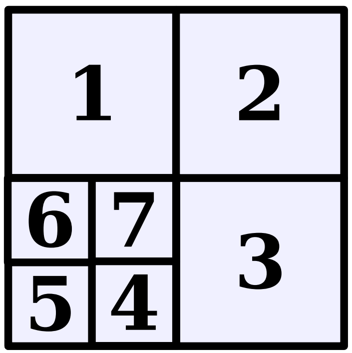

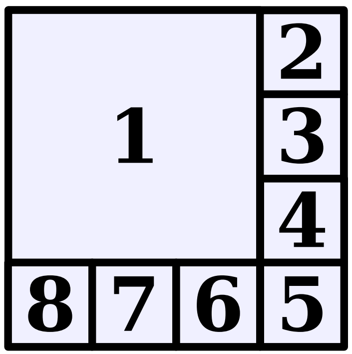

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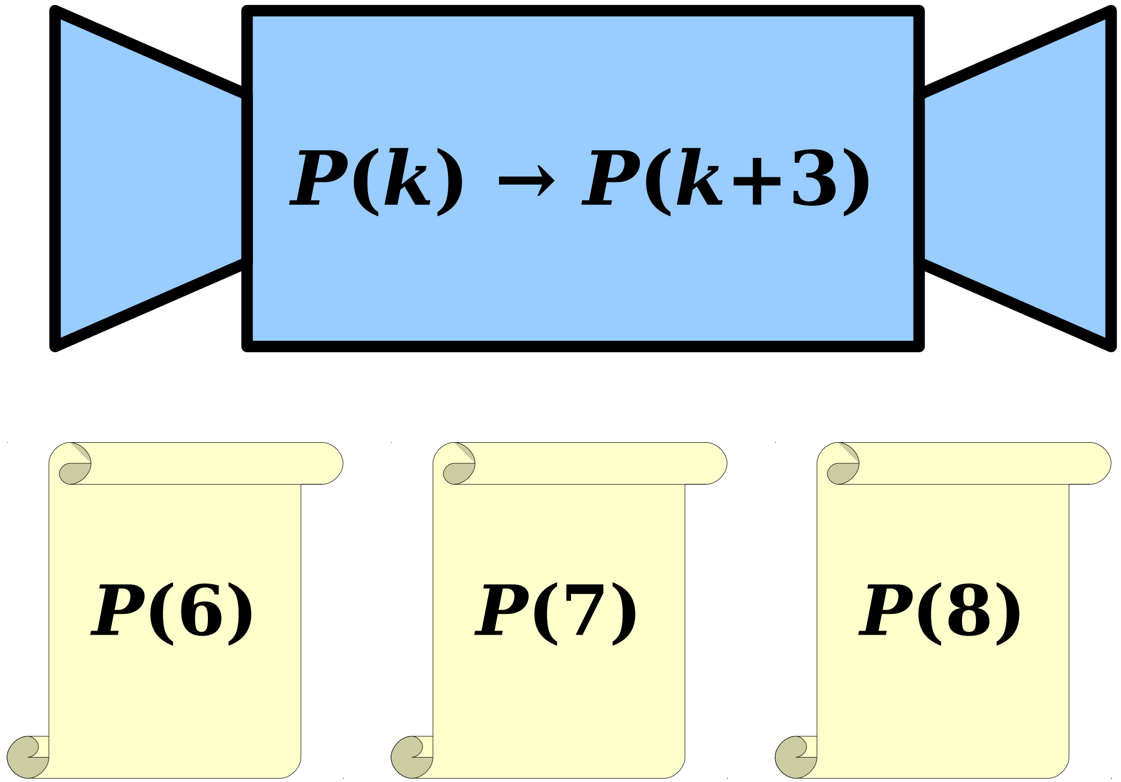
- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our "induction machine" analogy:



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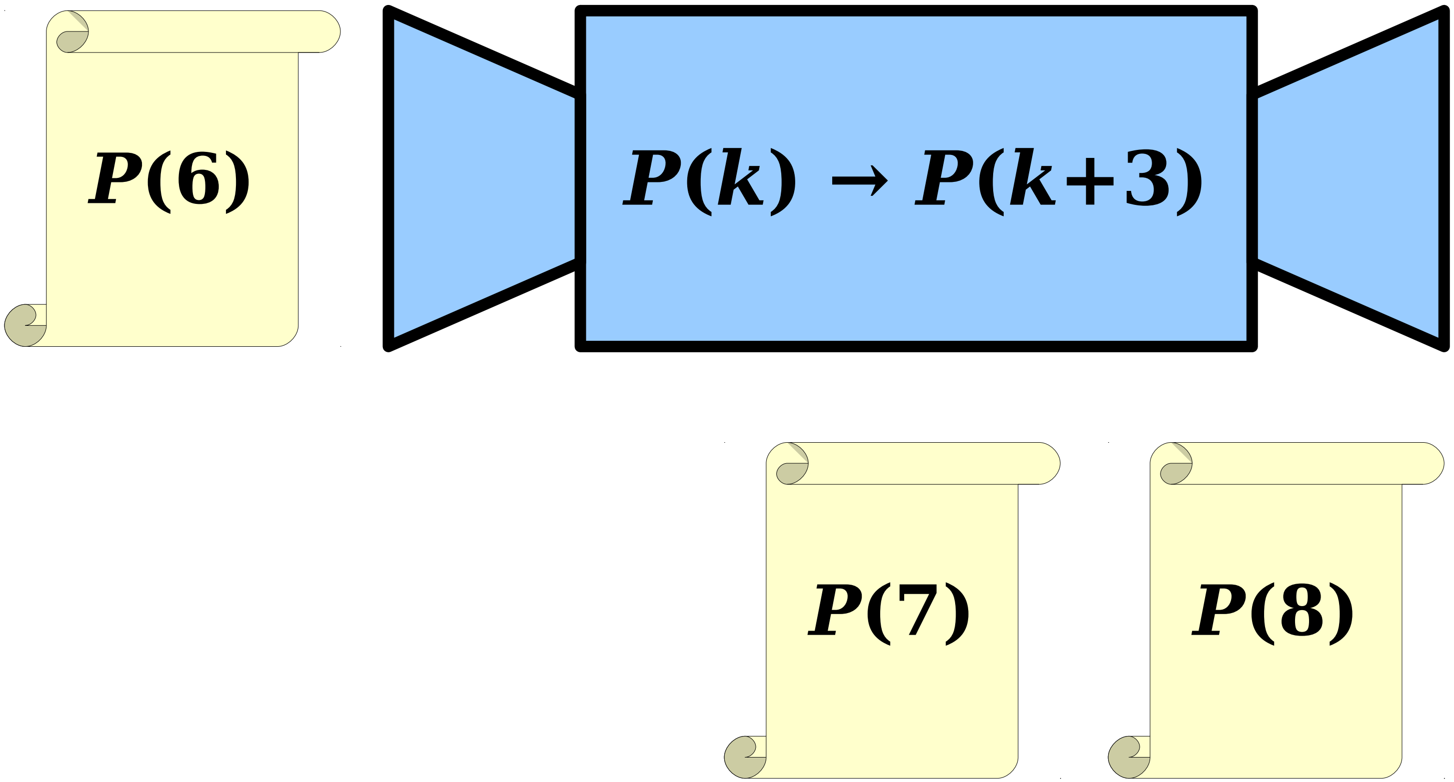

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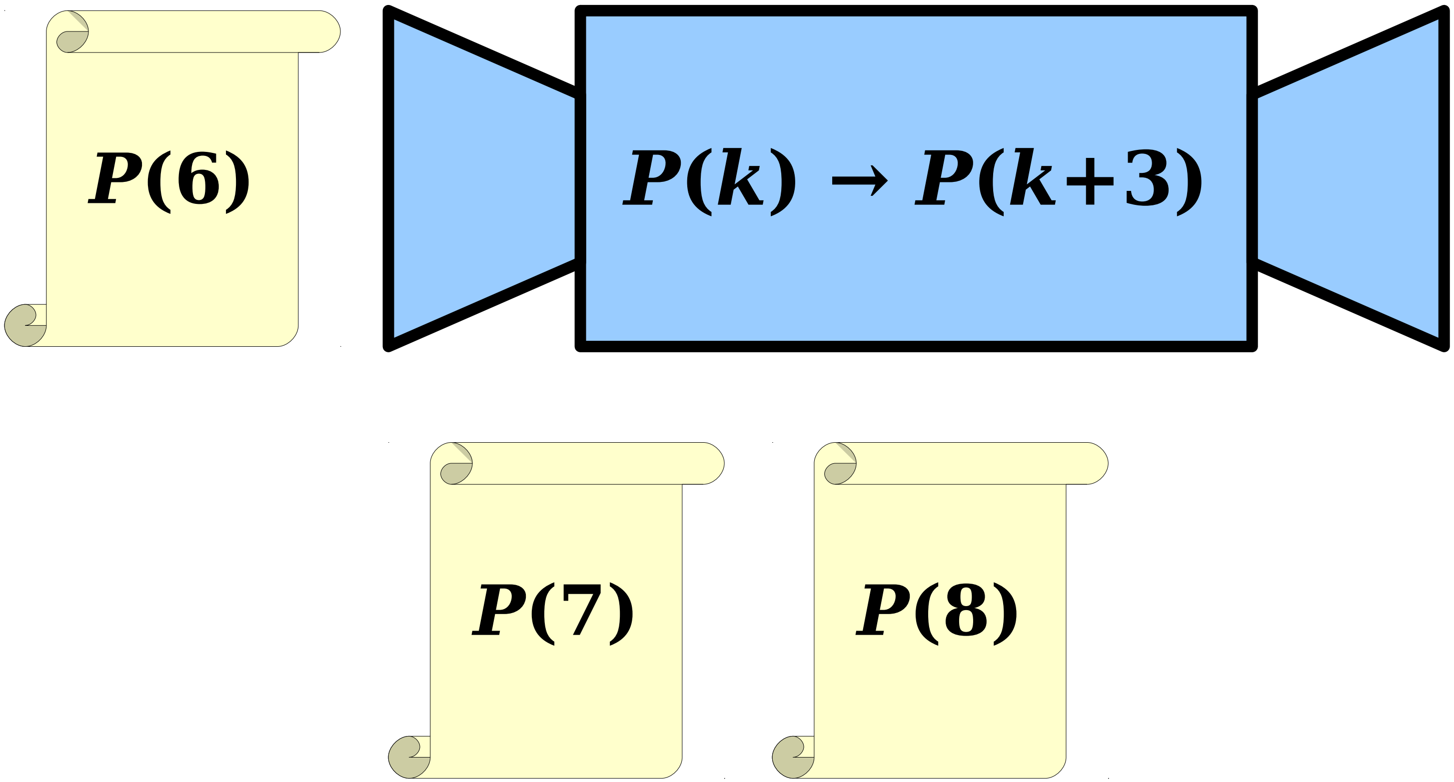

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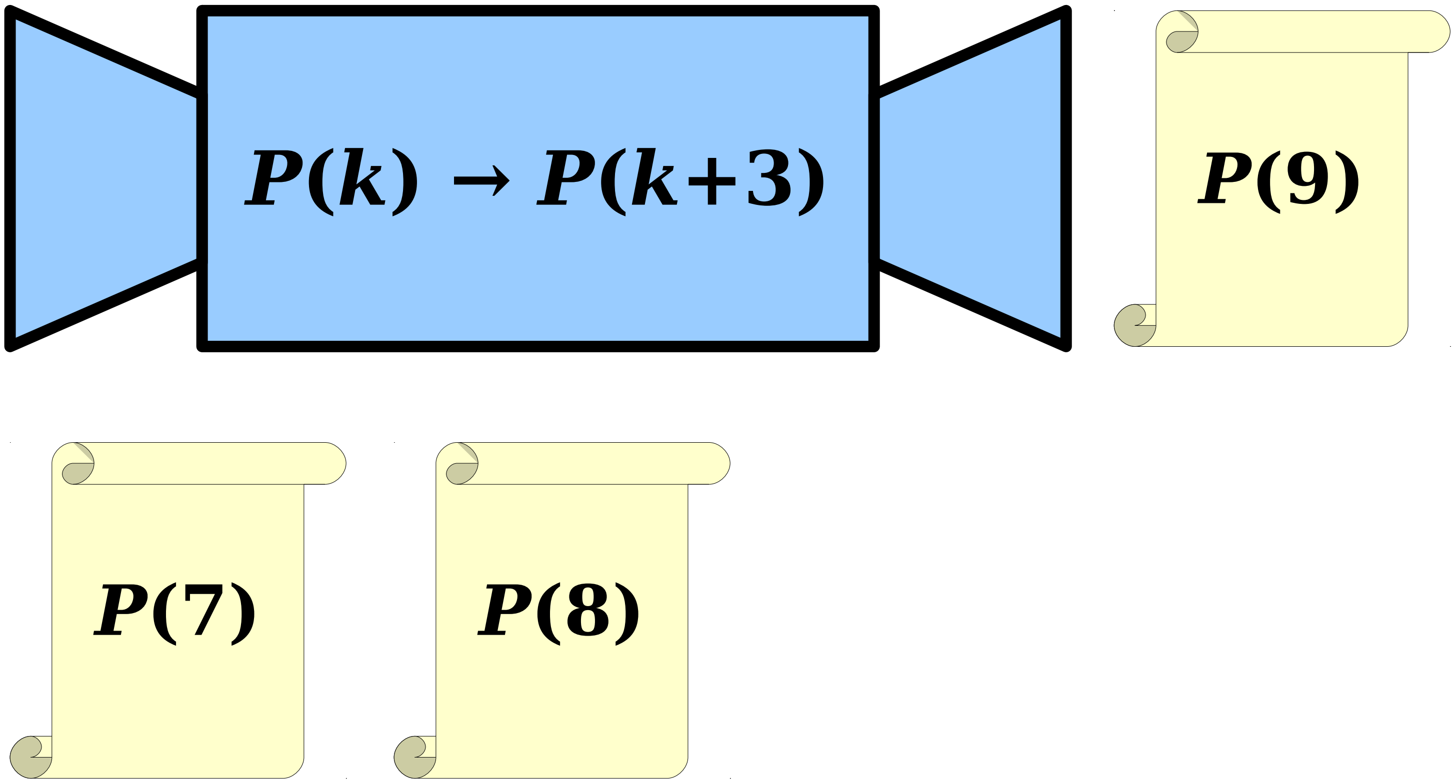

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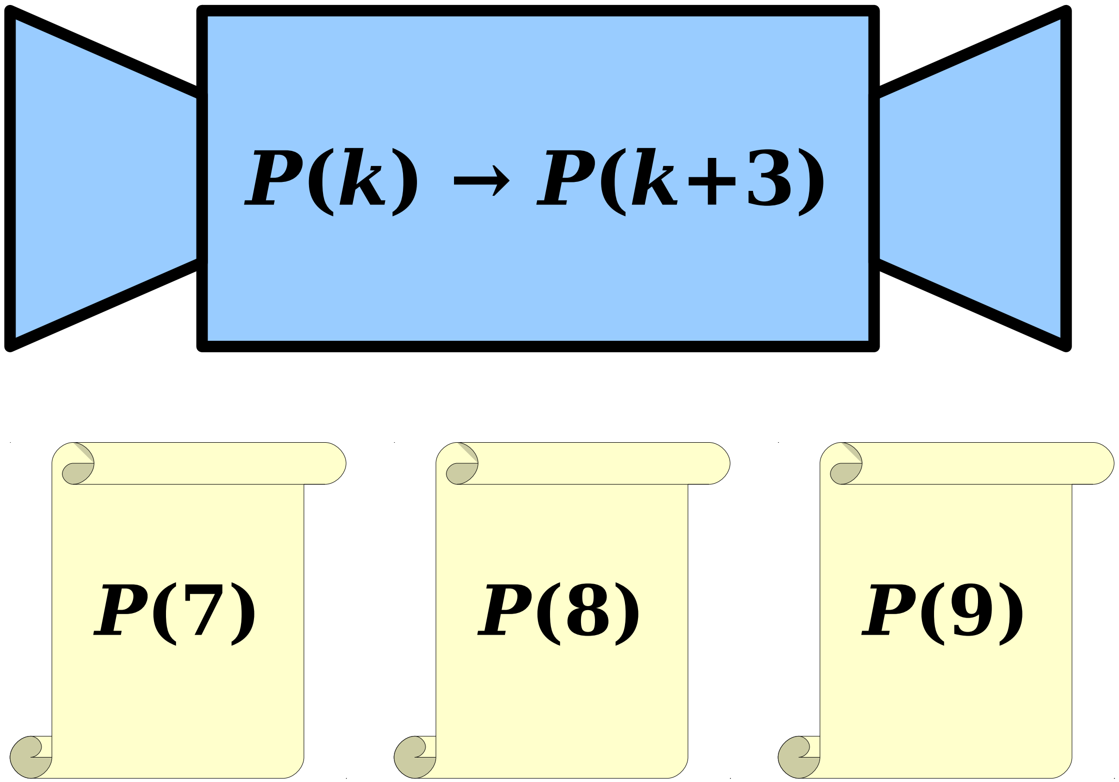

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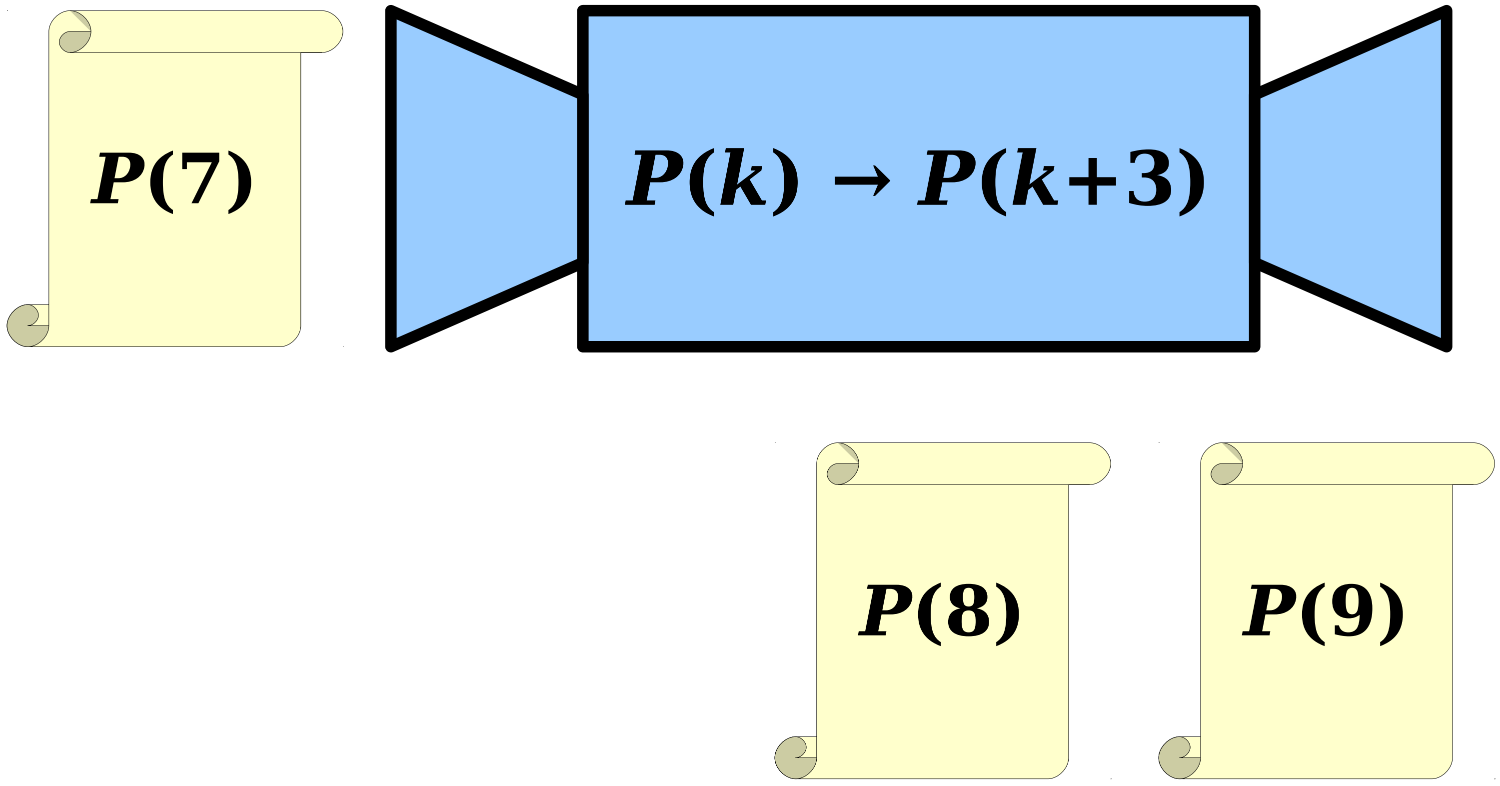

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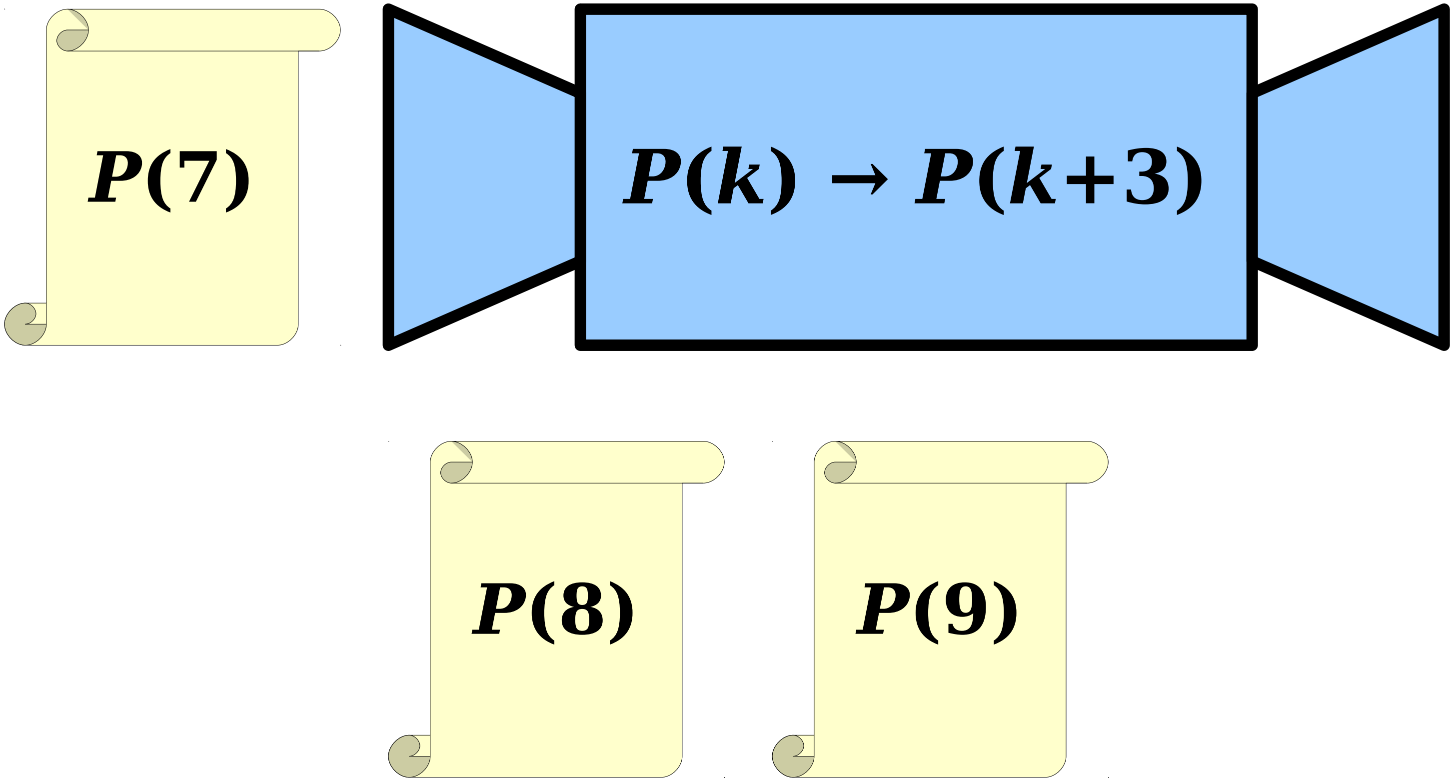

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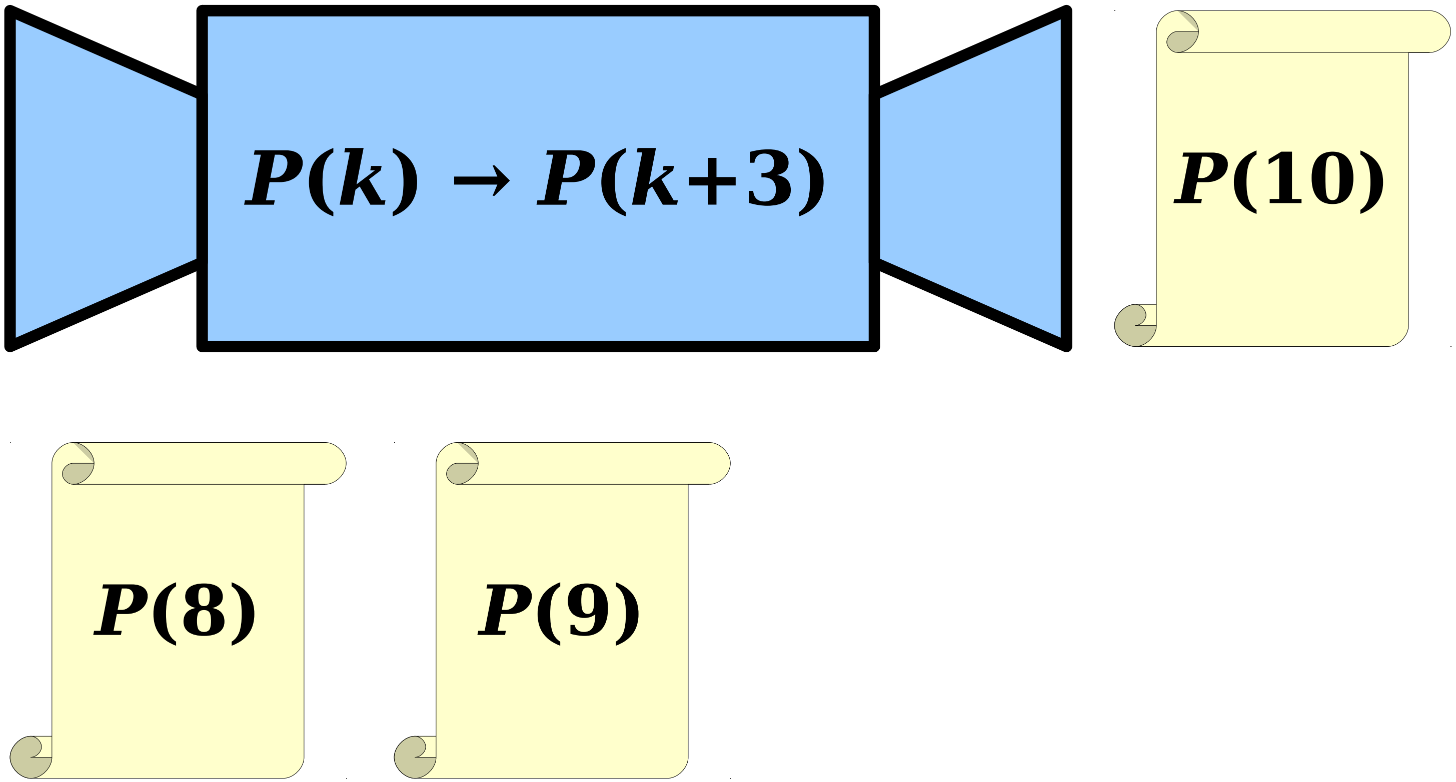

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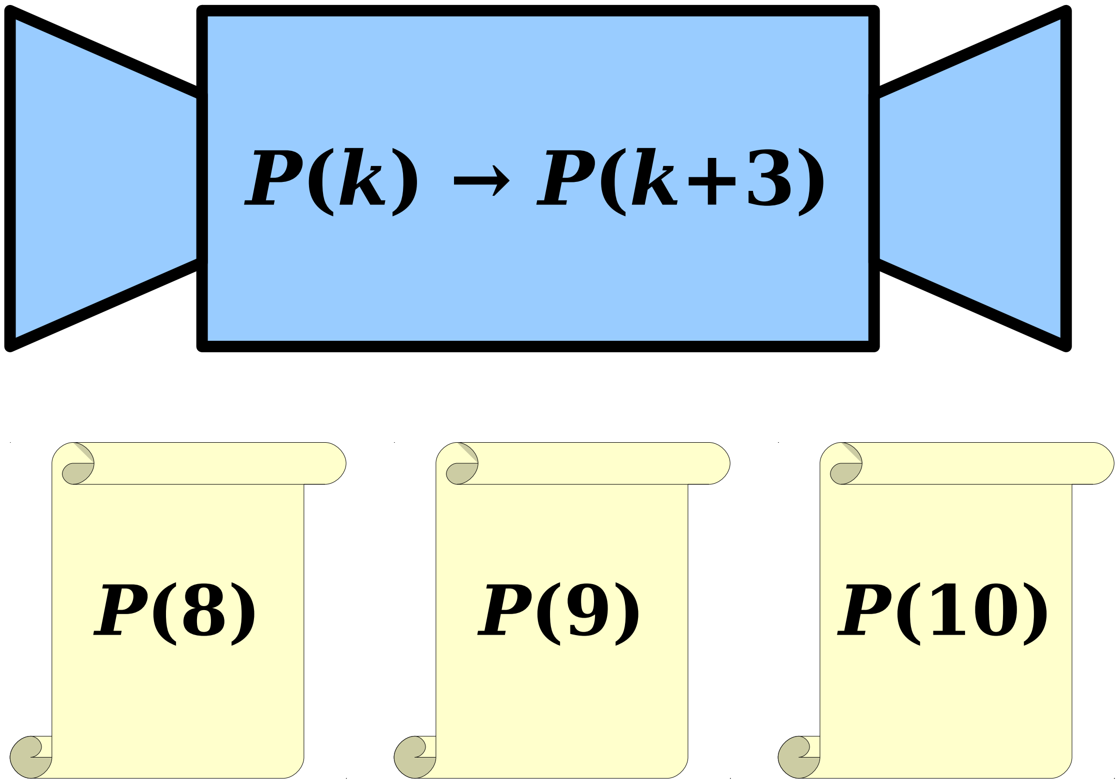

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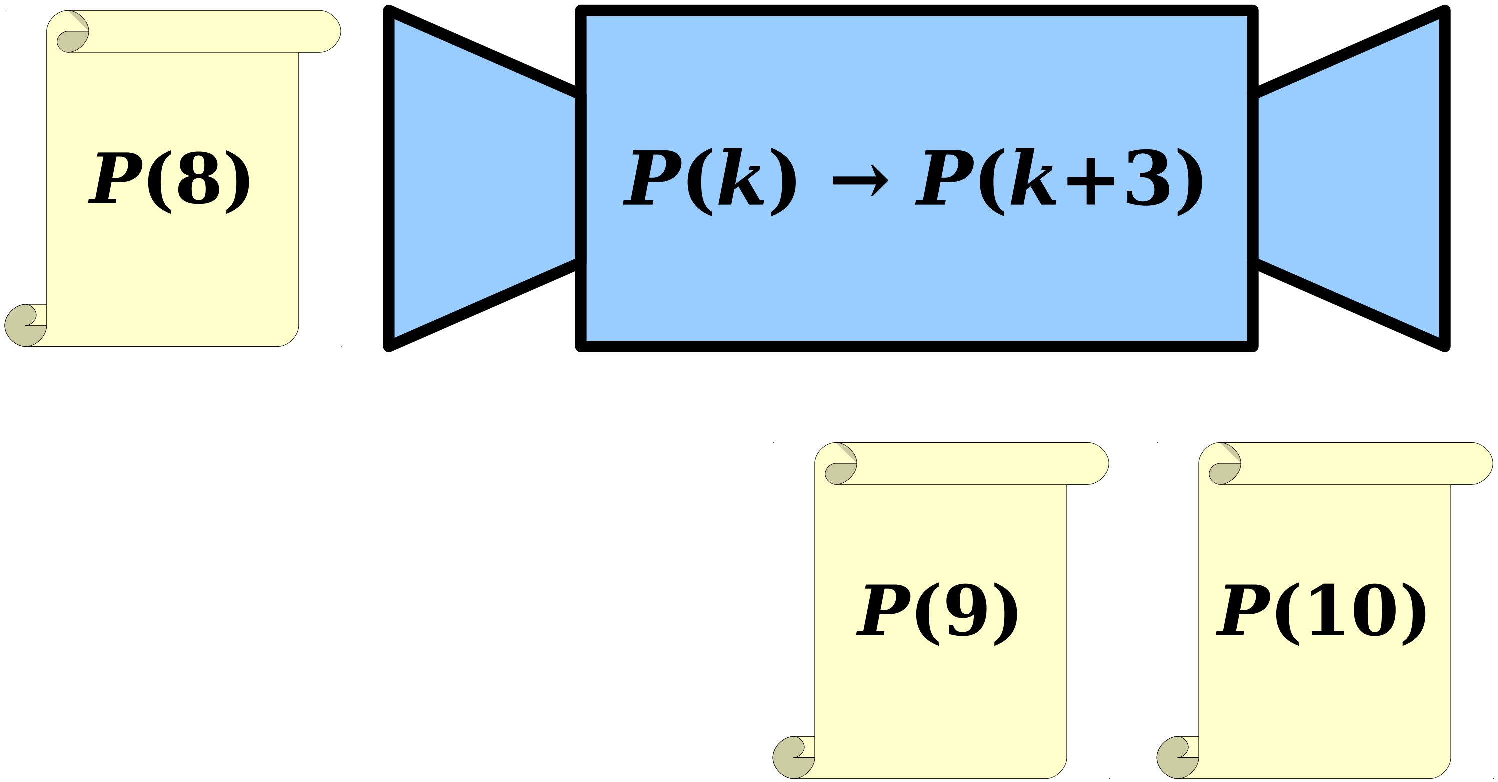

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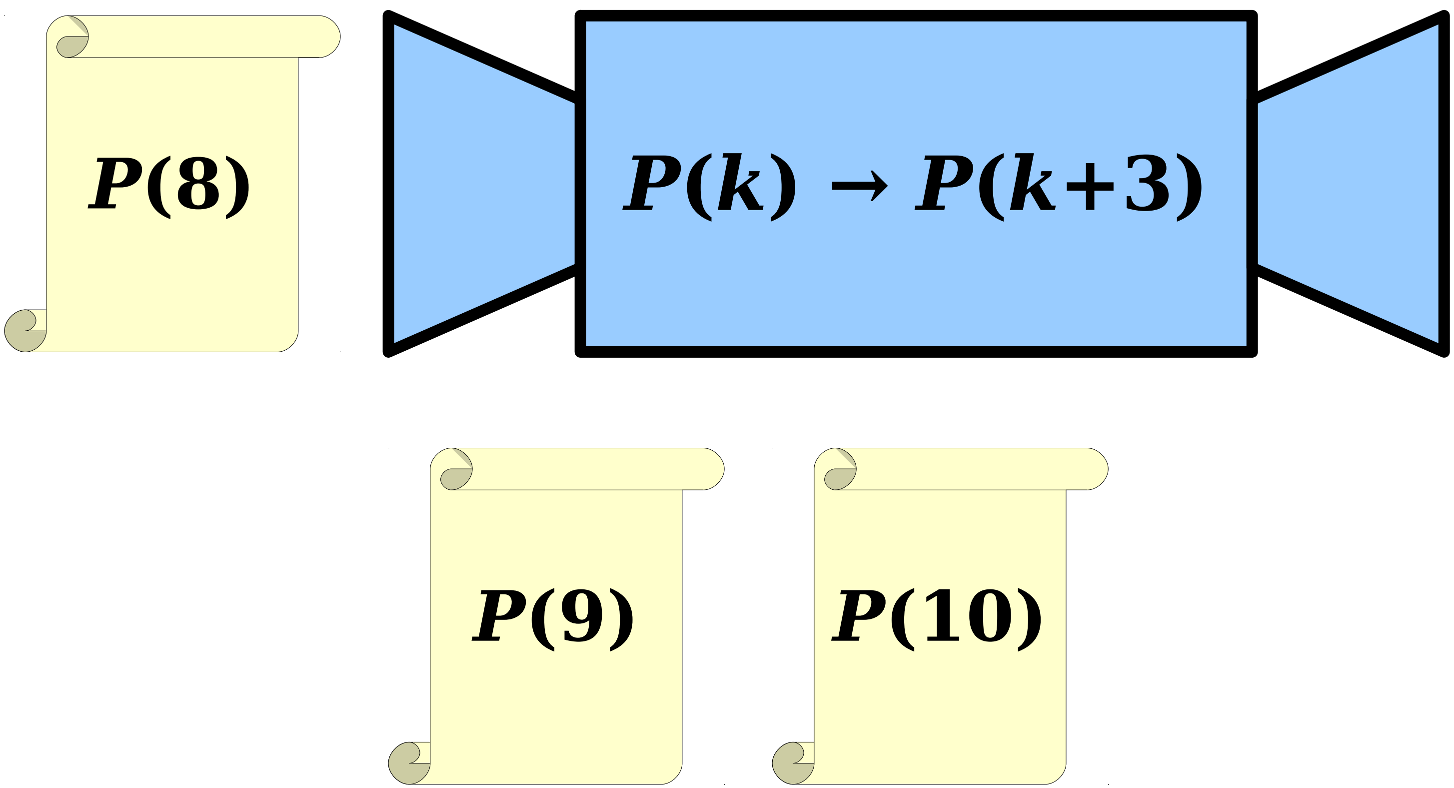

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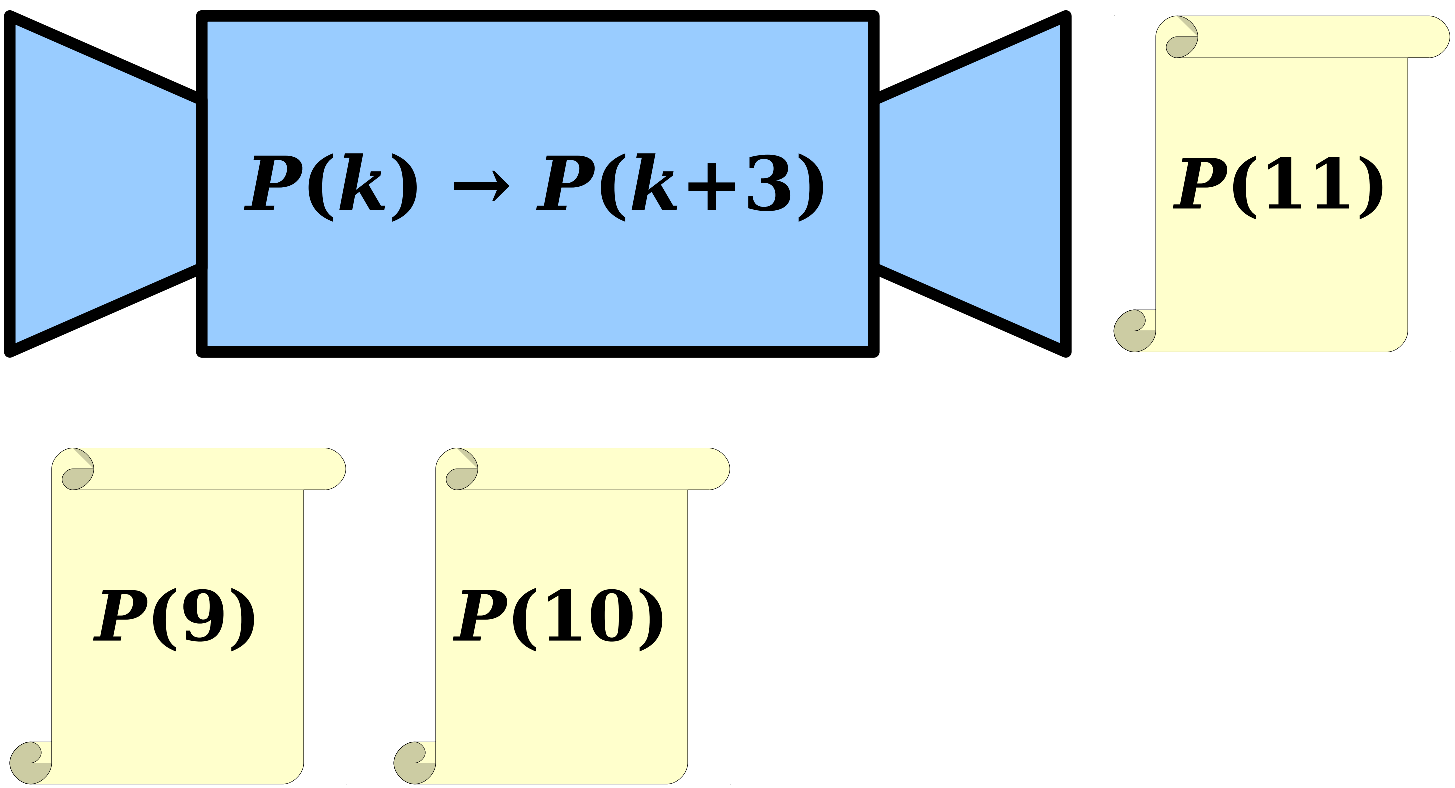

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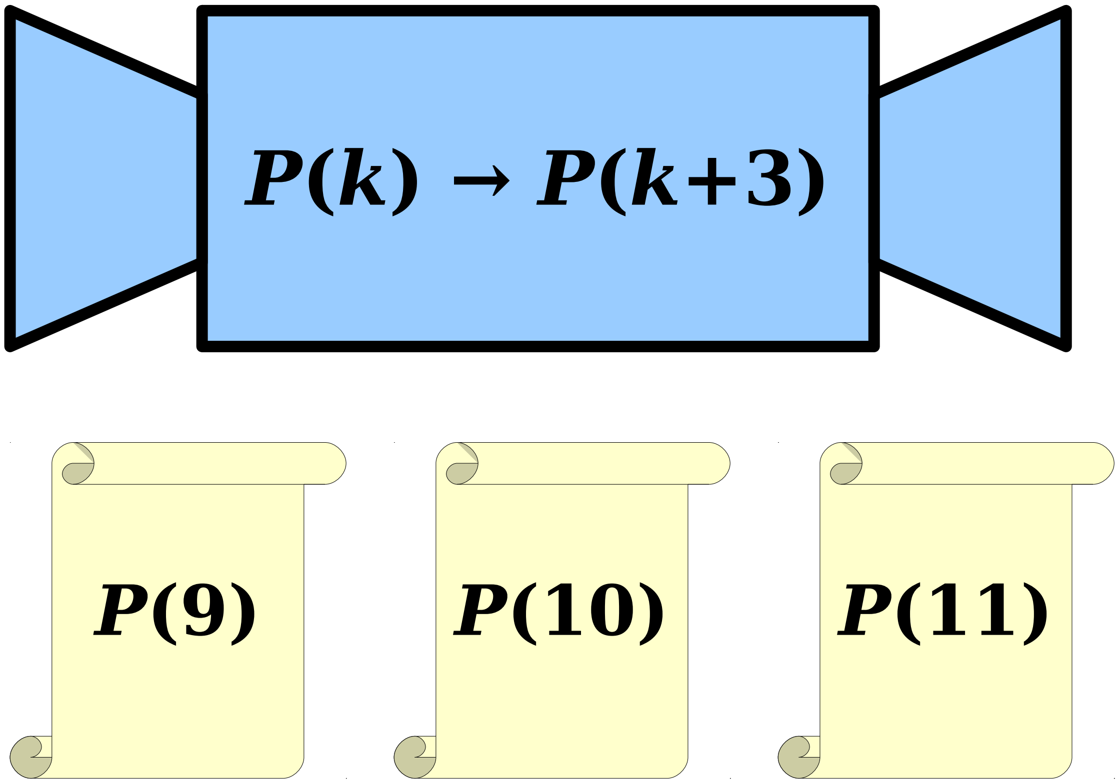


Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!
  - We won't require that you prove you've covered everything, but it doesn't hurt to double-check!

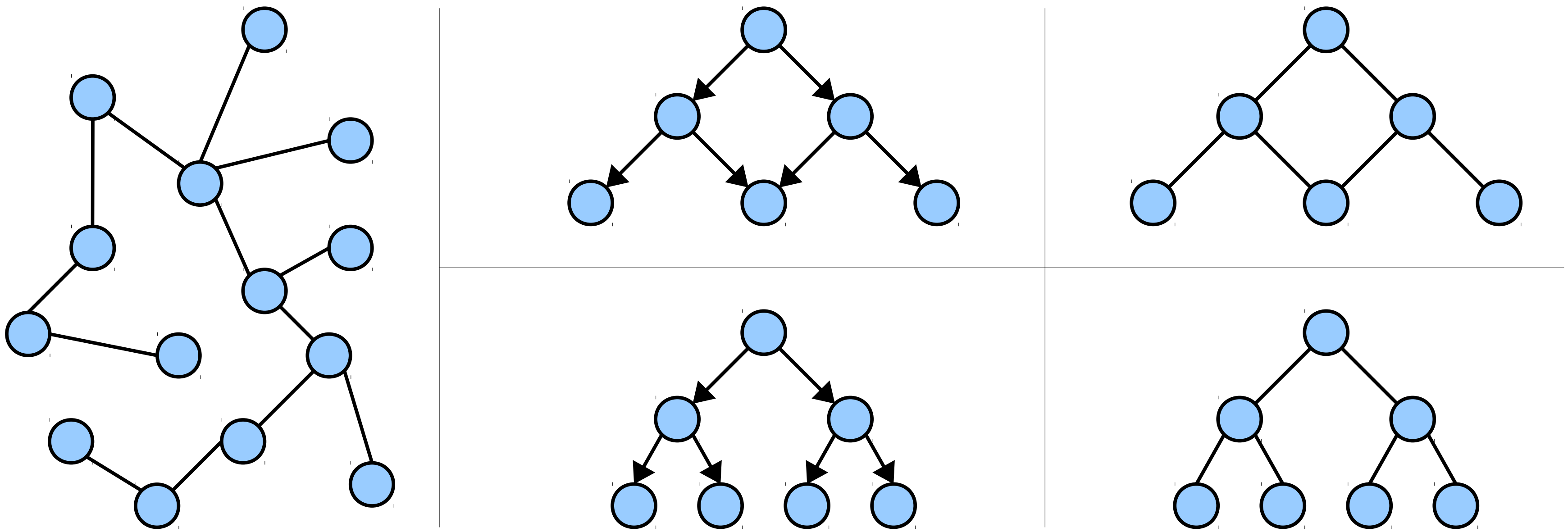
More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

A Special Type of Graph: ***Trees***

- A ***tree*** is a connected, nonempty graph with no simple cycles.

According to the above definition of trees, how many of these graphs are trees?



Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then a number.

Trees

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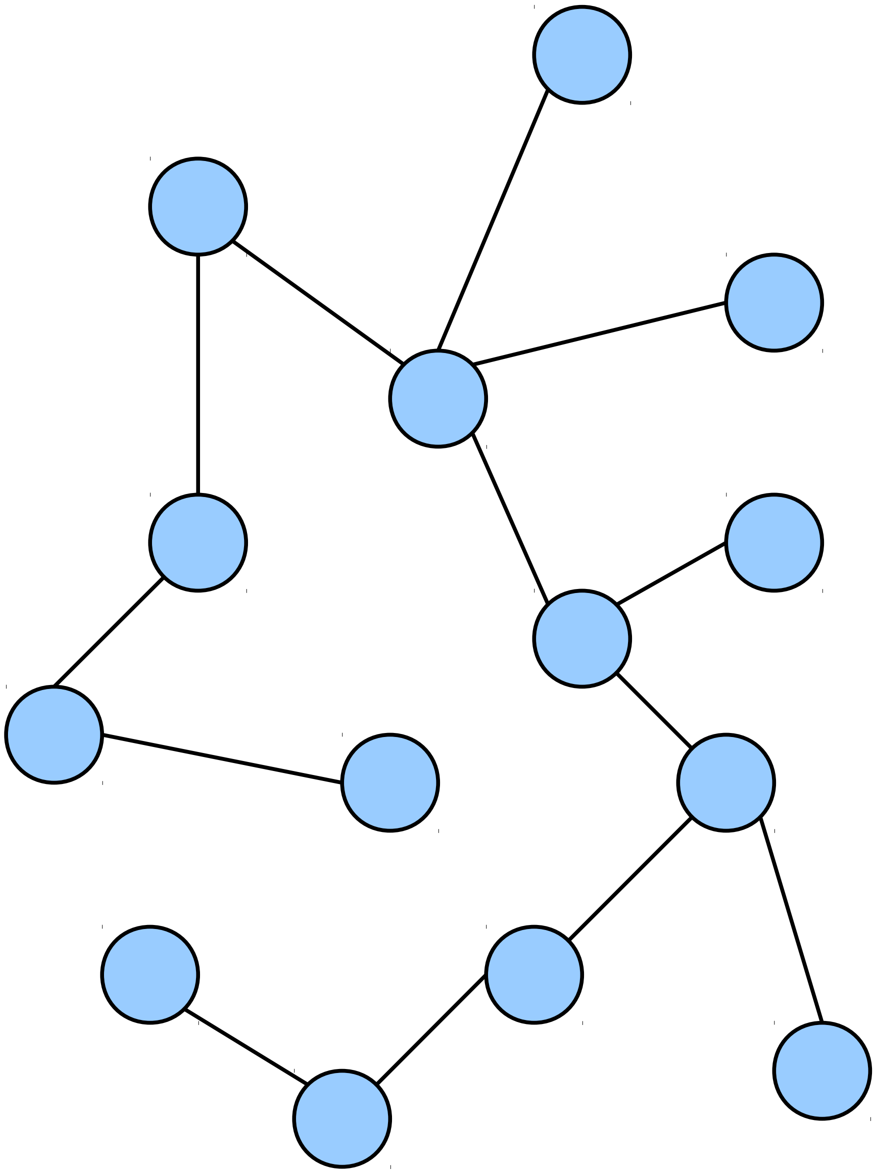
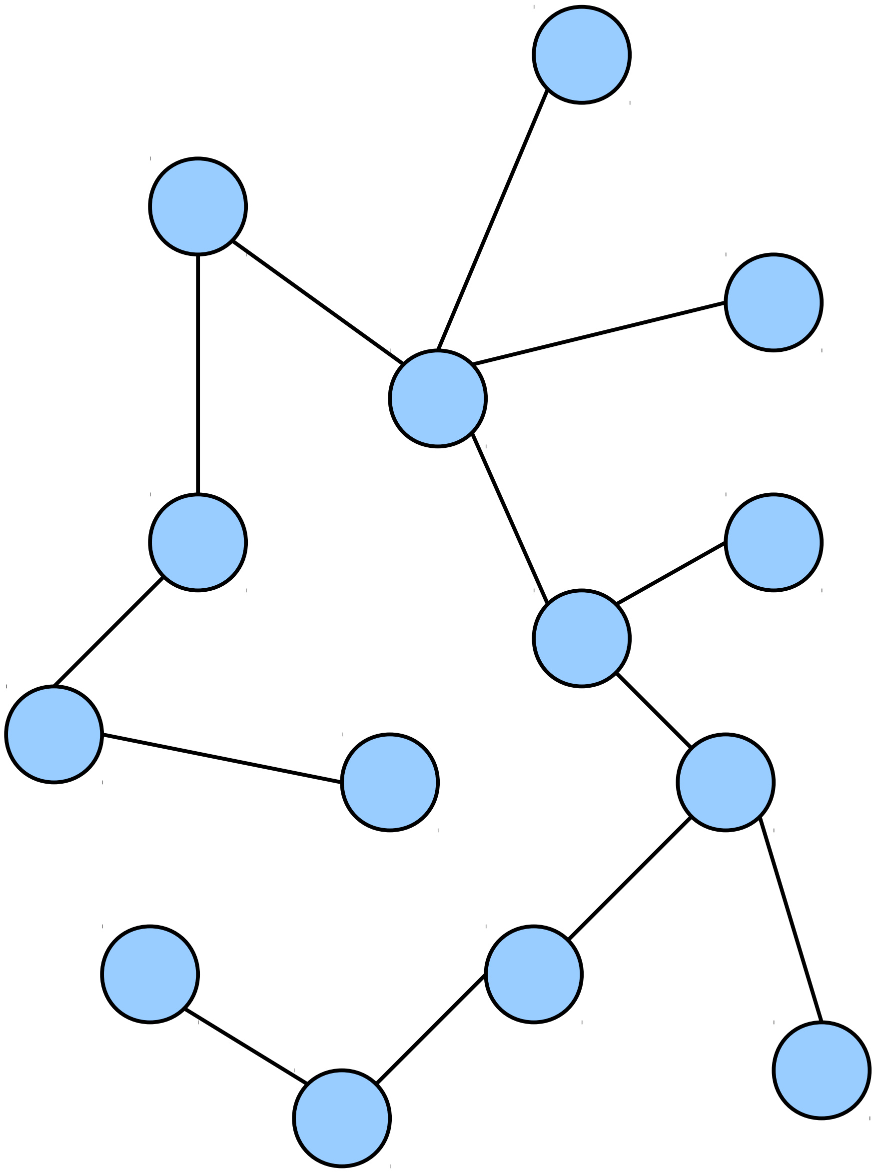


Trees

- A ***tree*** is a connected, nonempty graph with no simple cycles.
- Trees have tons of nice properties:
  - They're ***maximally acyclic*** (adding any missing edge creates a simple cycle)



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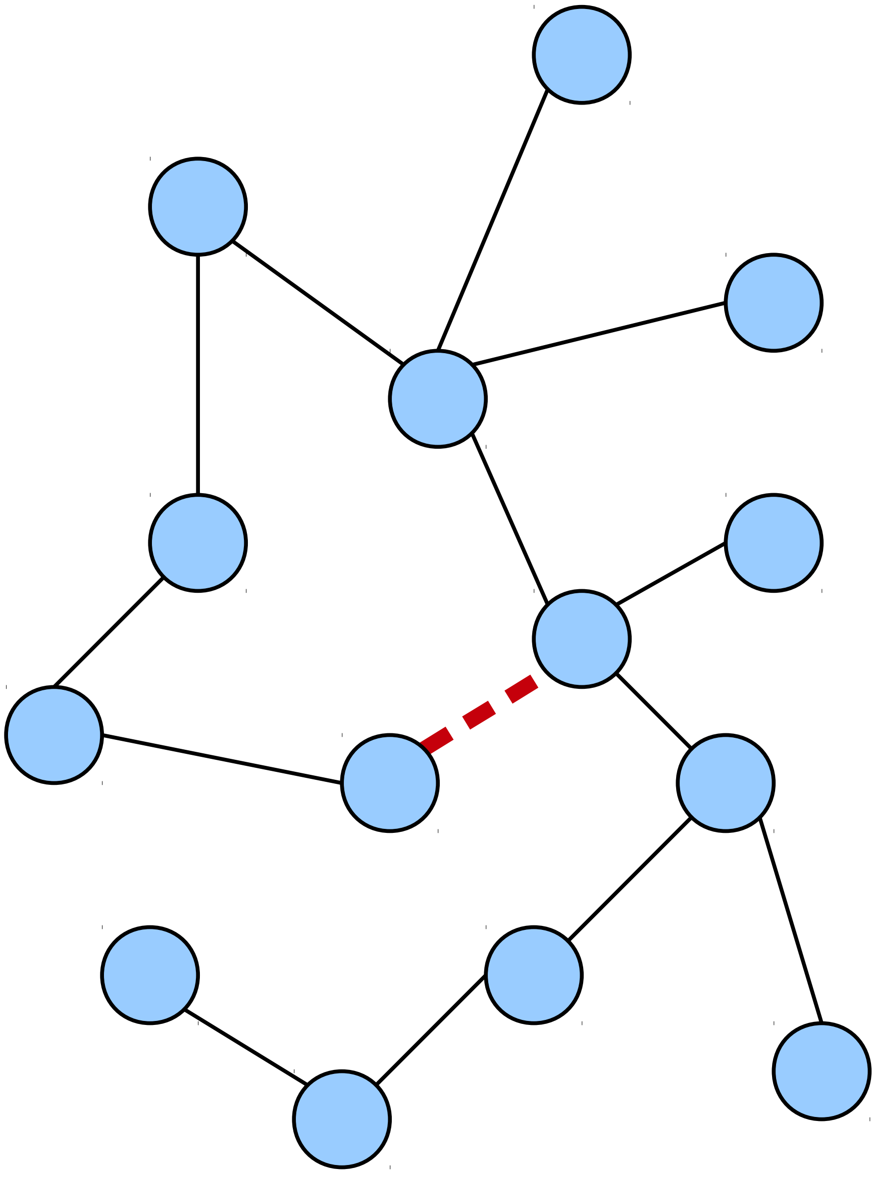

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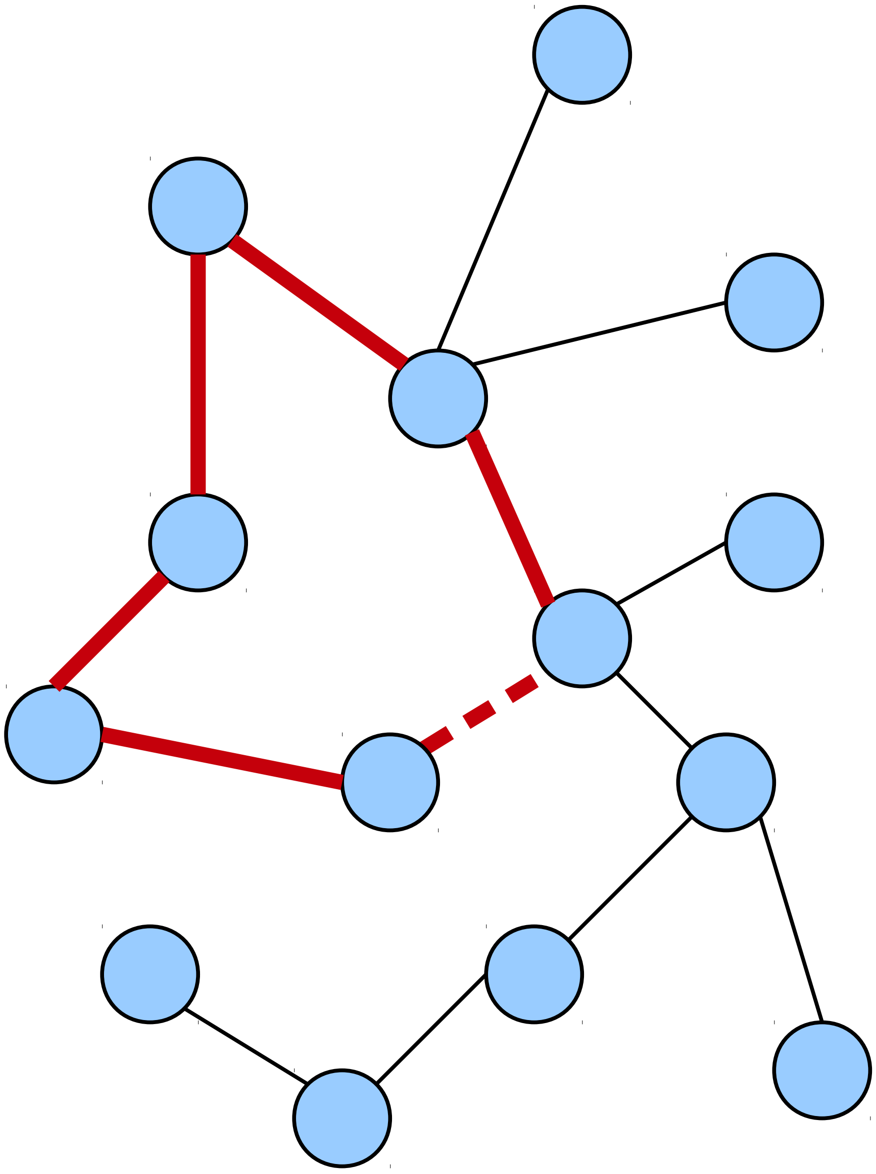

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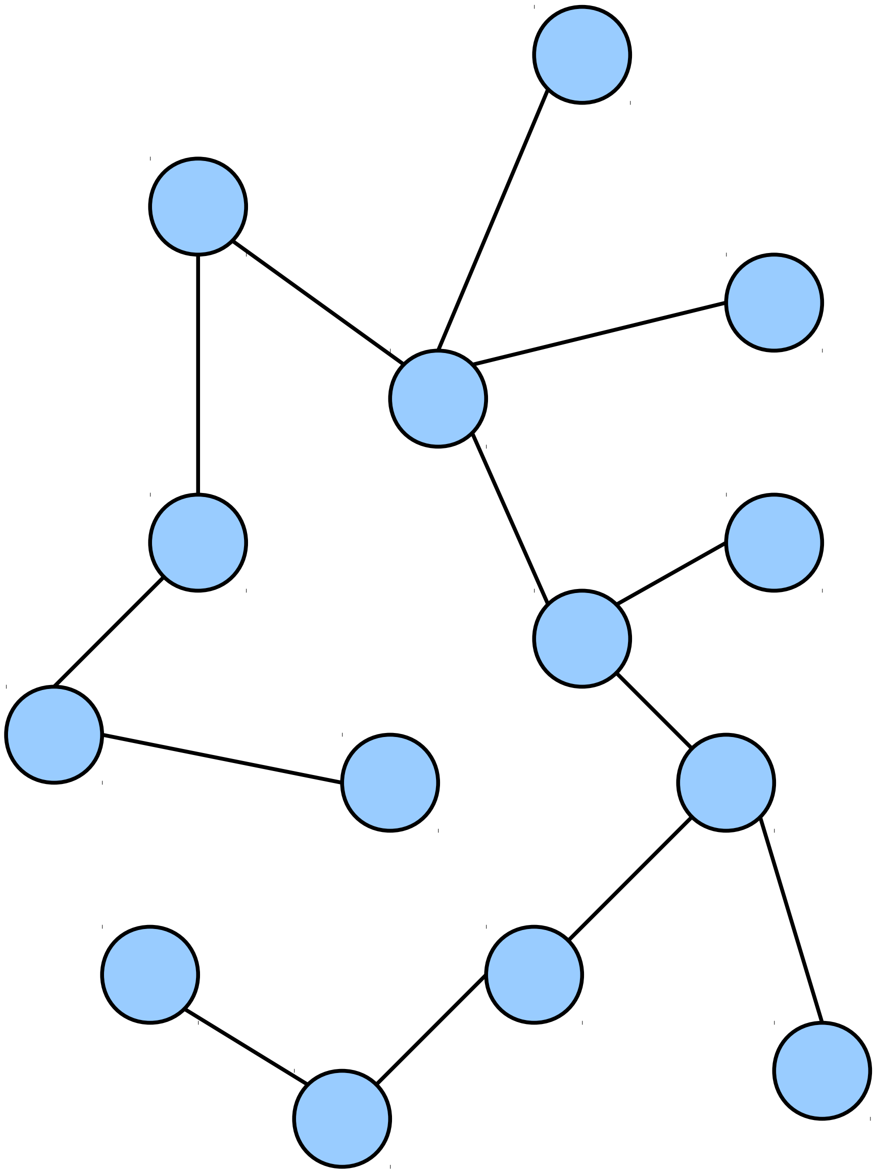

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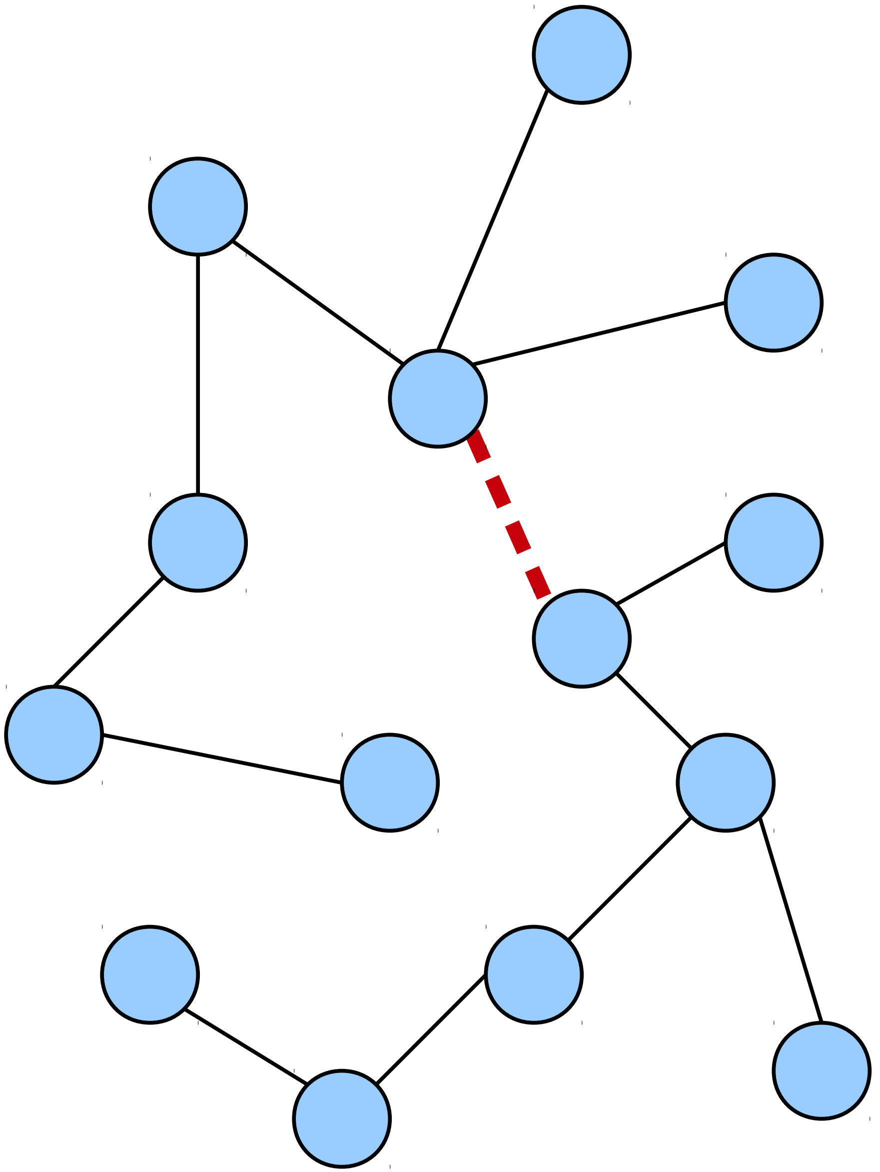

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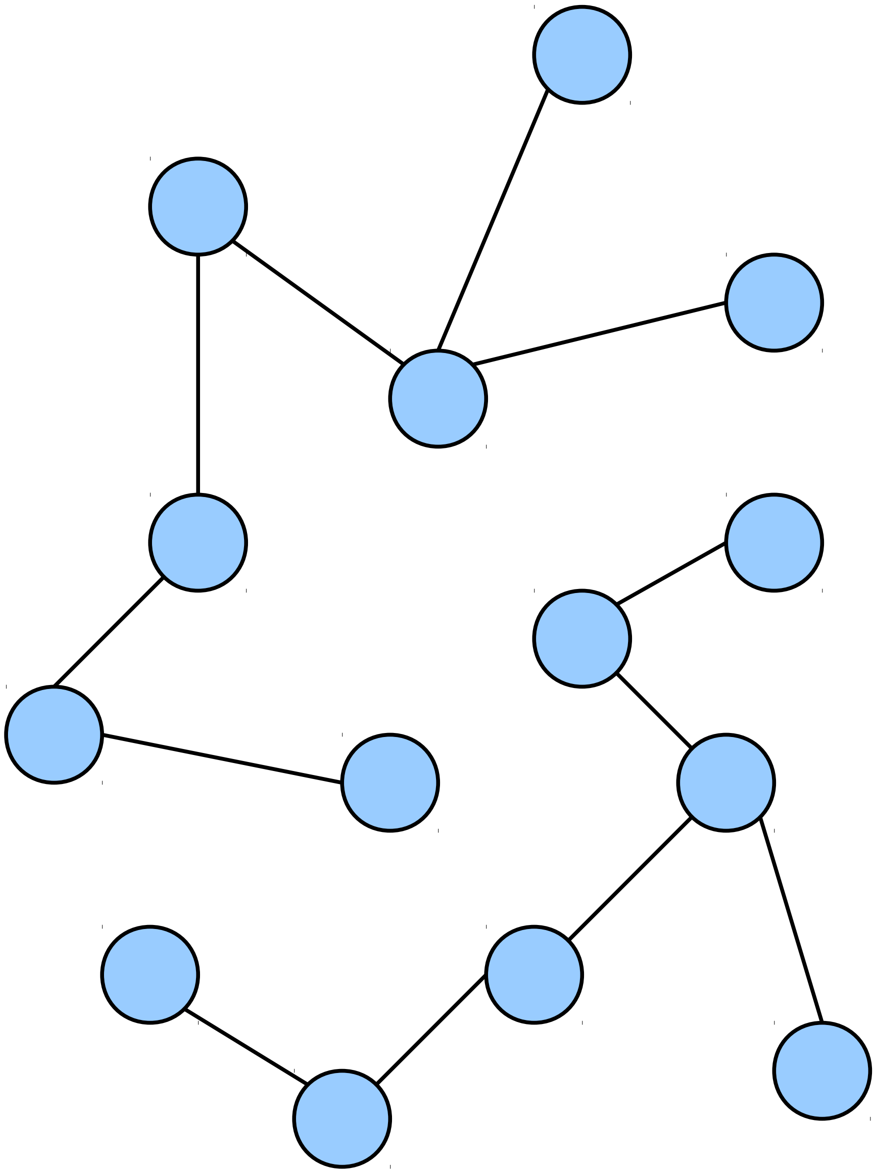

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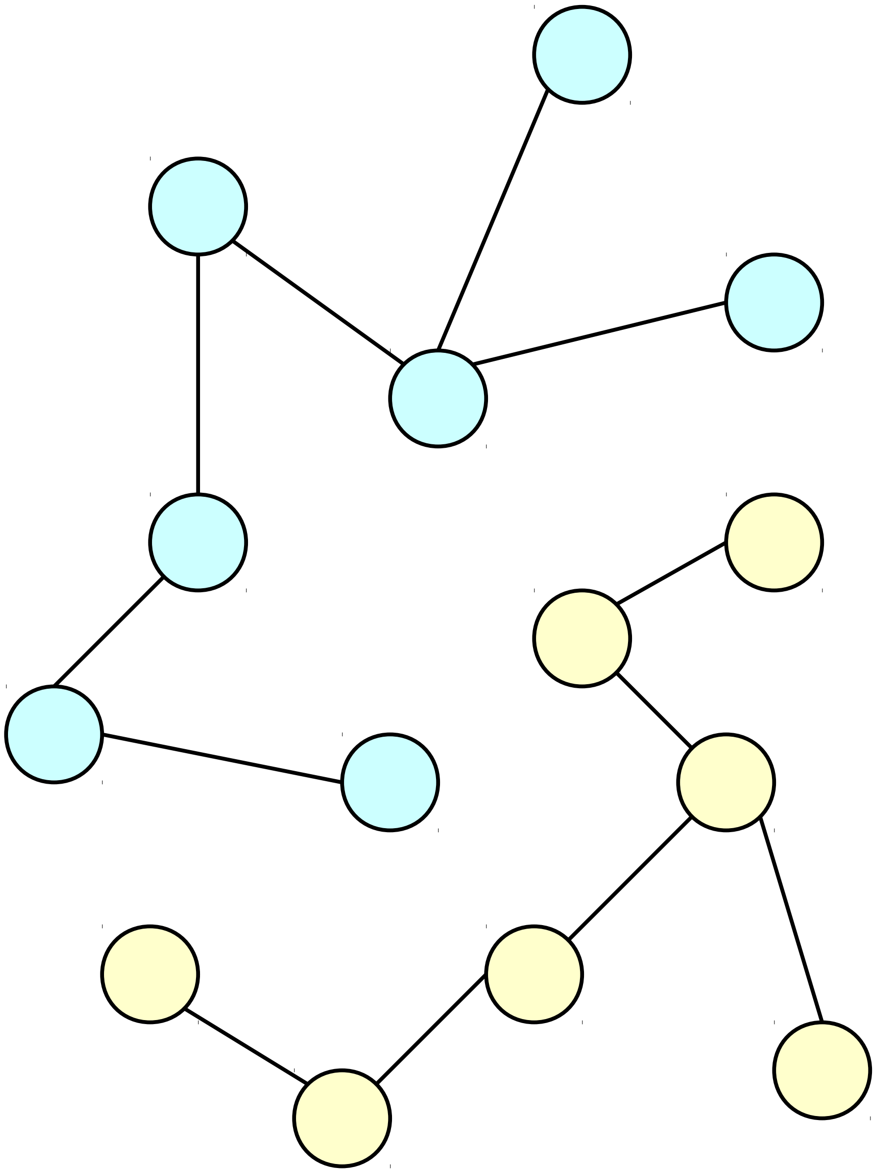

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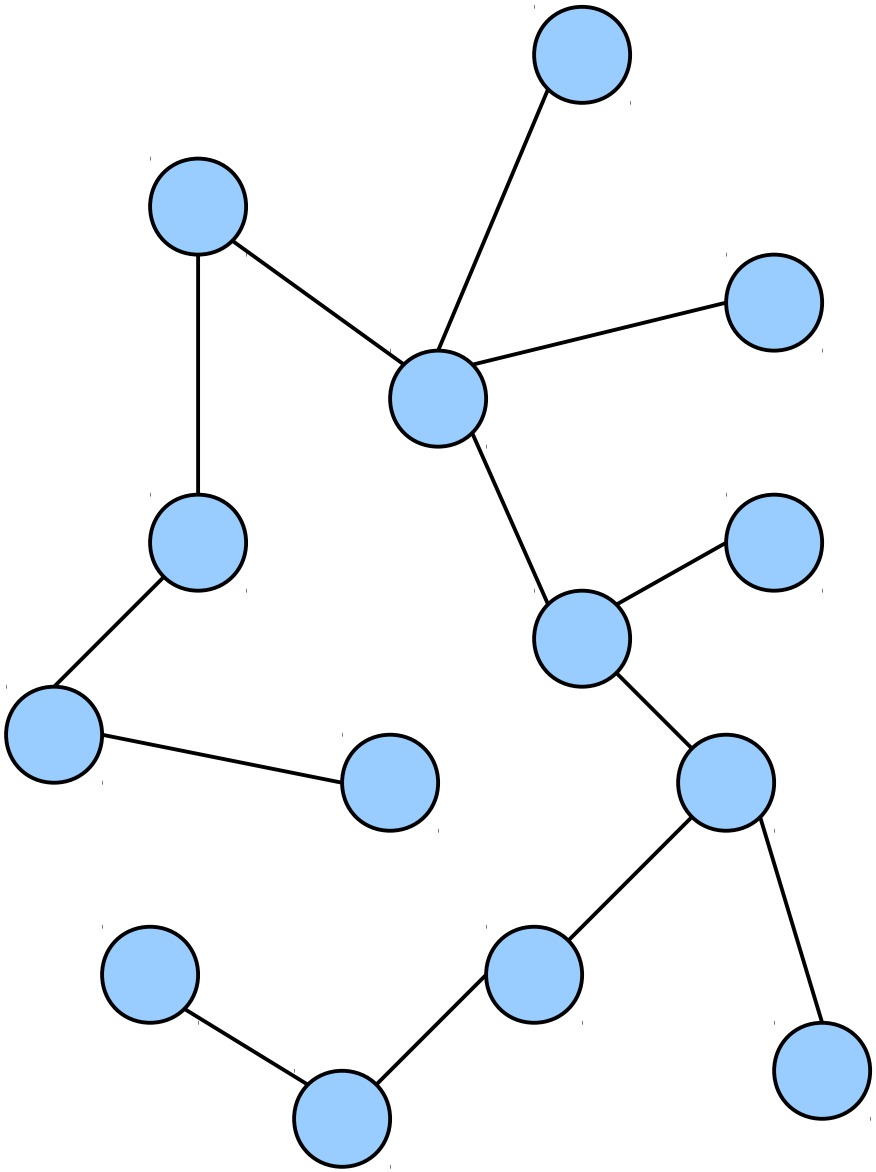
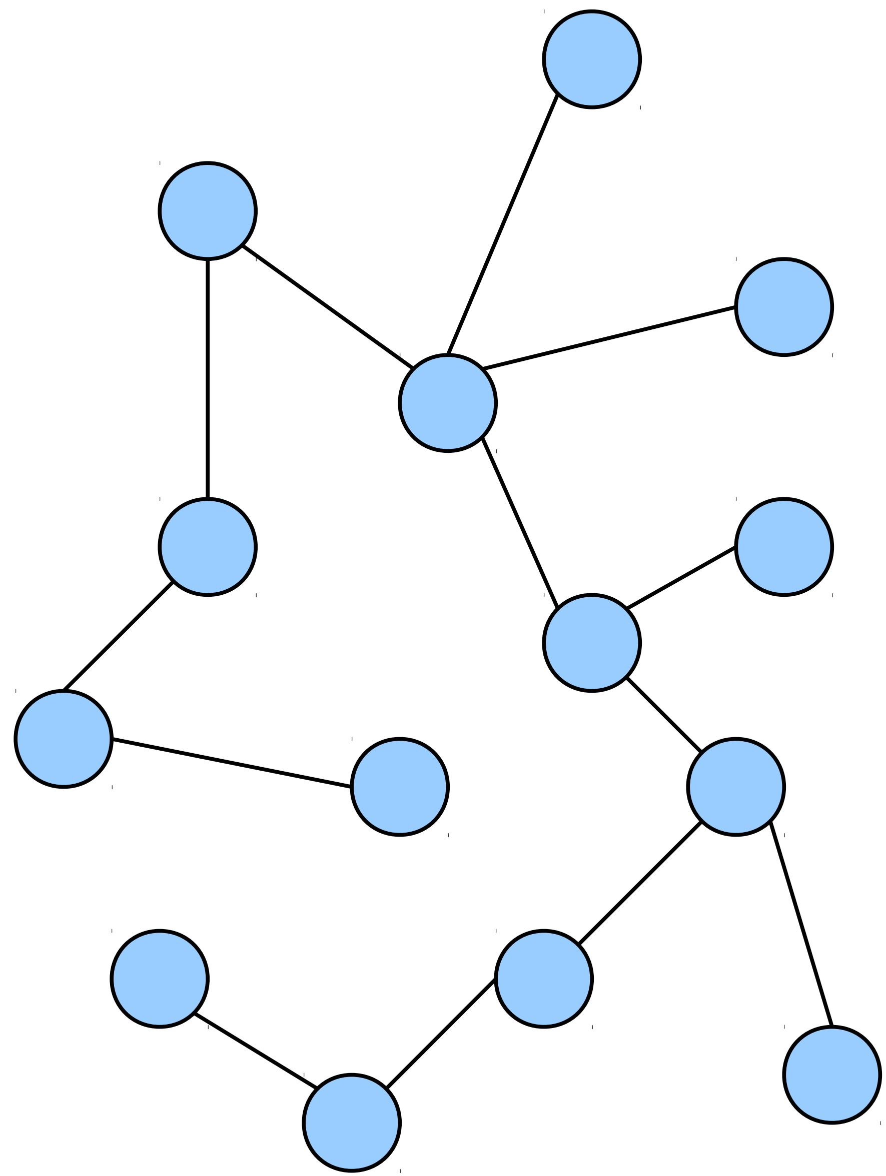
- Proofs of these results are in the course reader if you're interested. They're also great exercises.



Trees

- ***Theorem:*** If T is a tree with at least two nodes, then deleting any edge from T splits T into two nonempty trees T_1 and T_2.
- ***Proof:*** Left as an exercise to the reader. ☺



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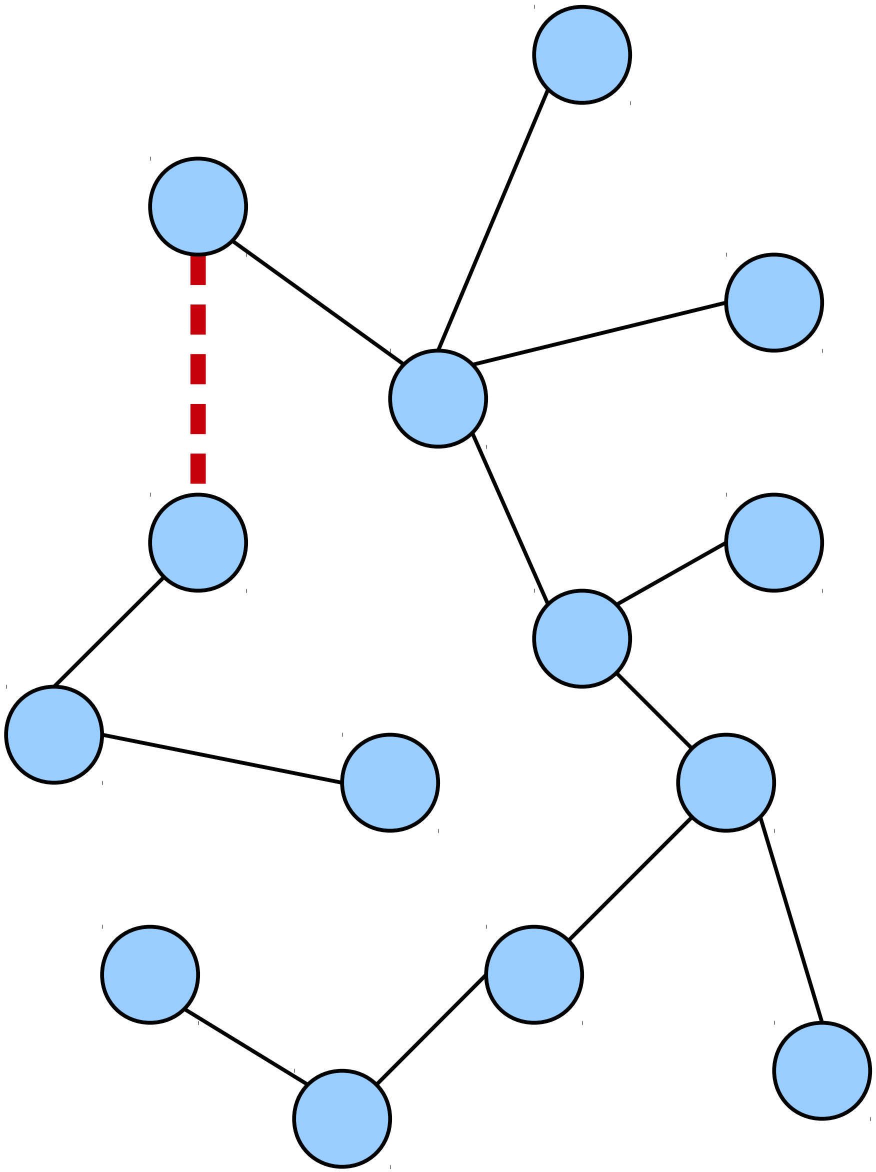

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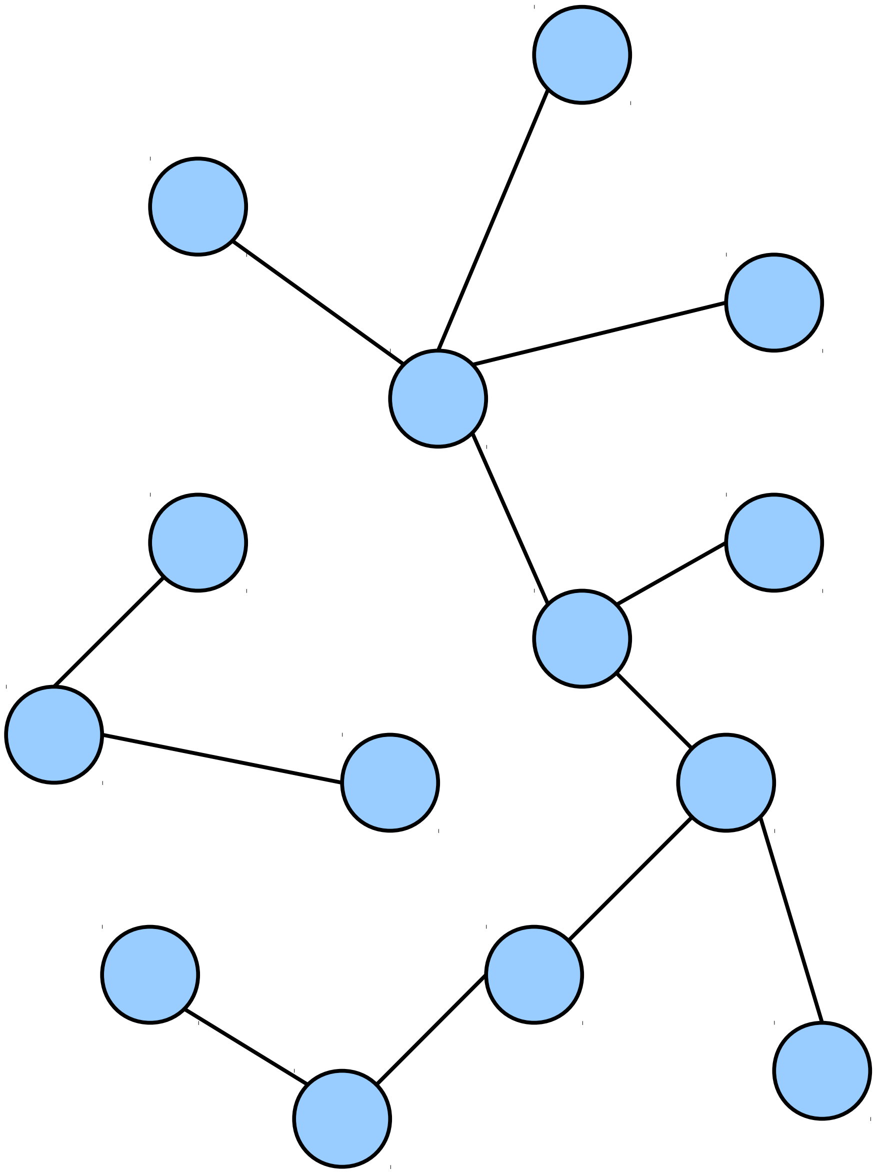

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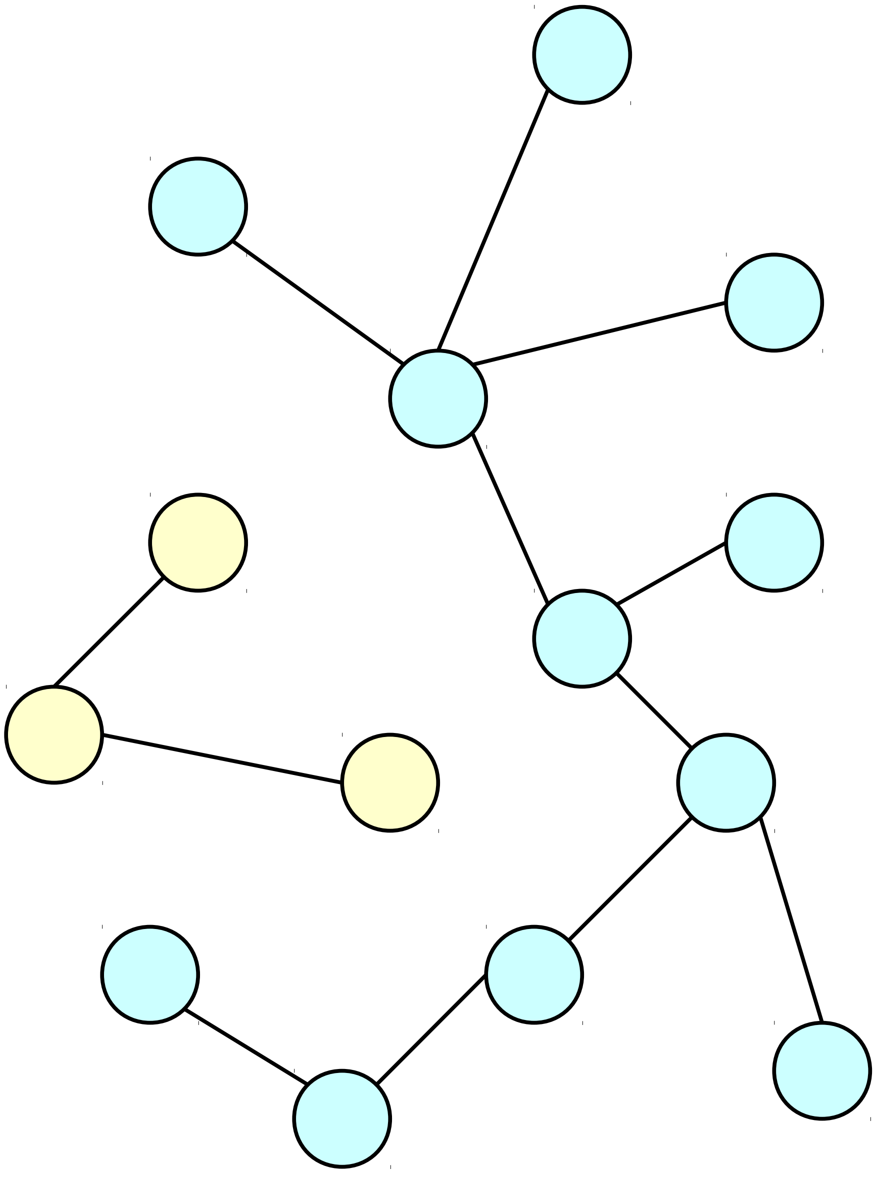

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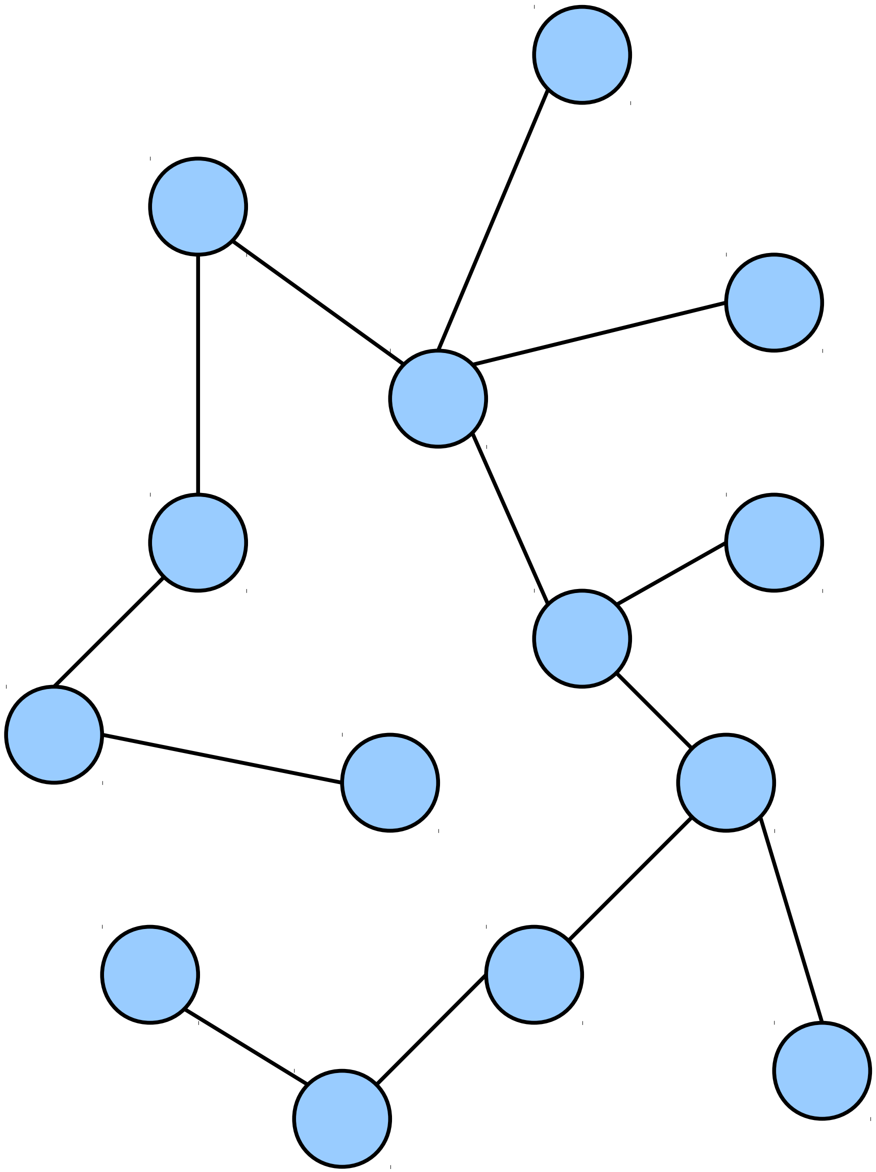

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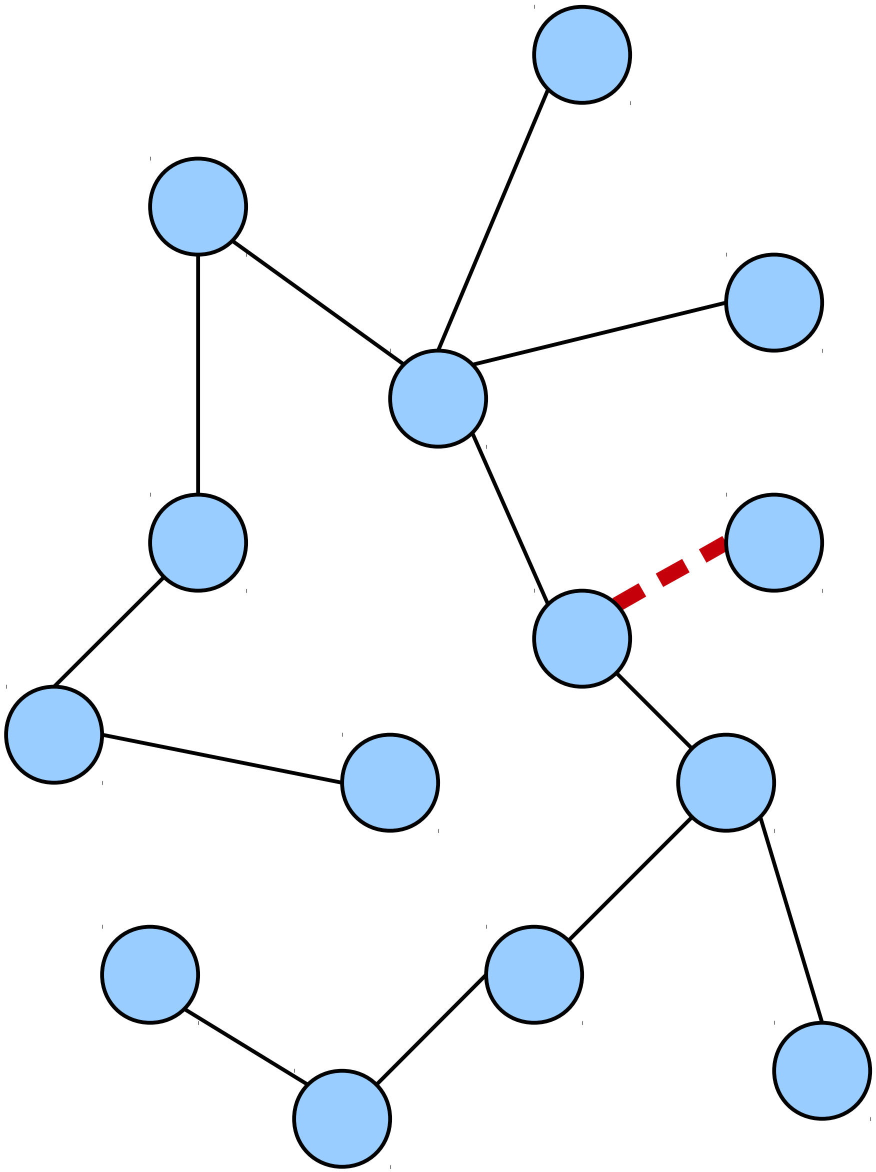

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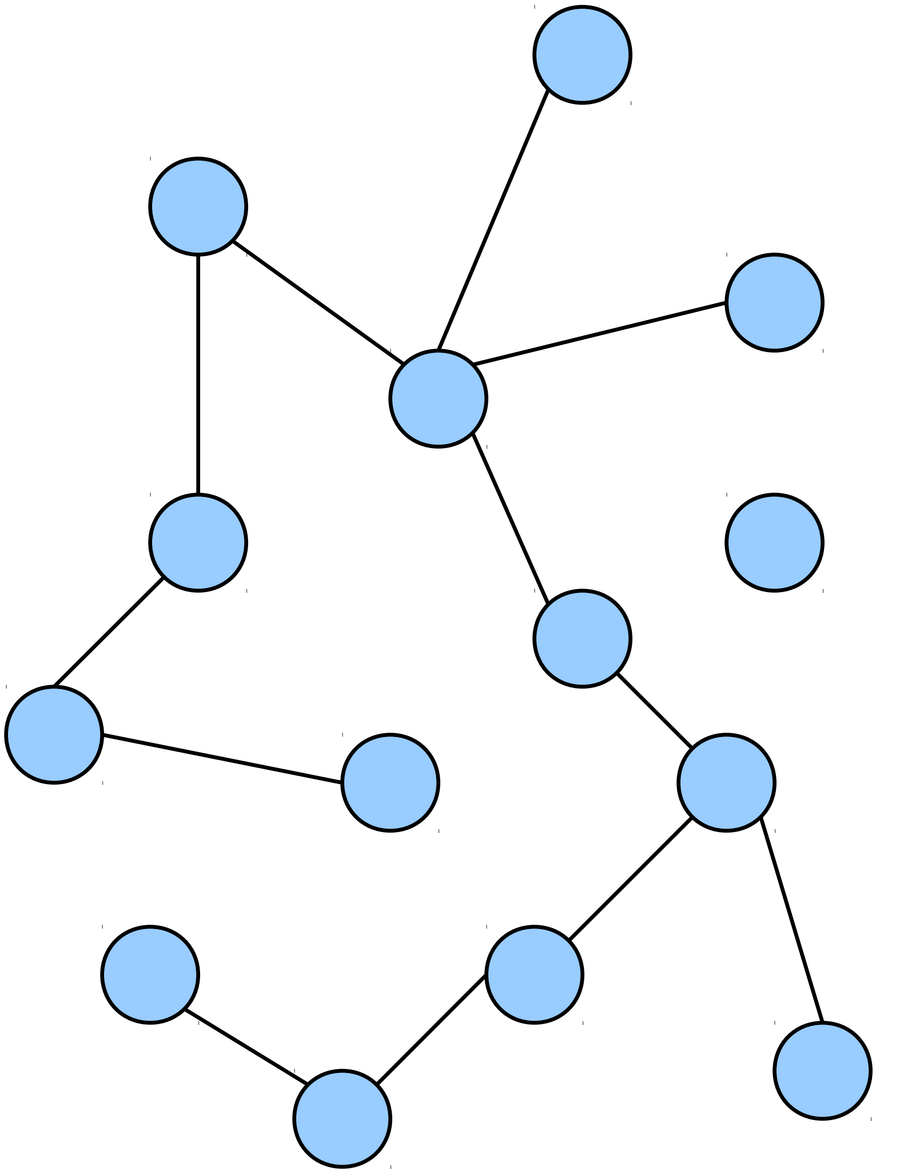

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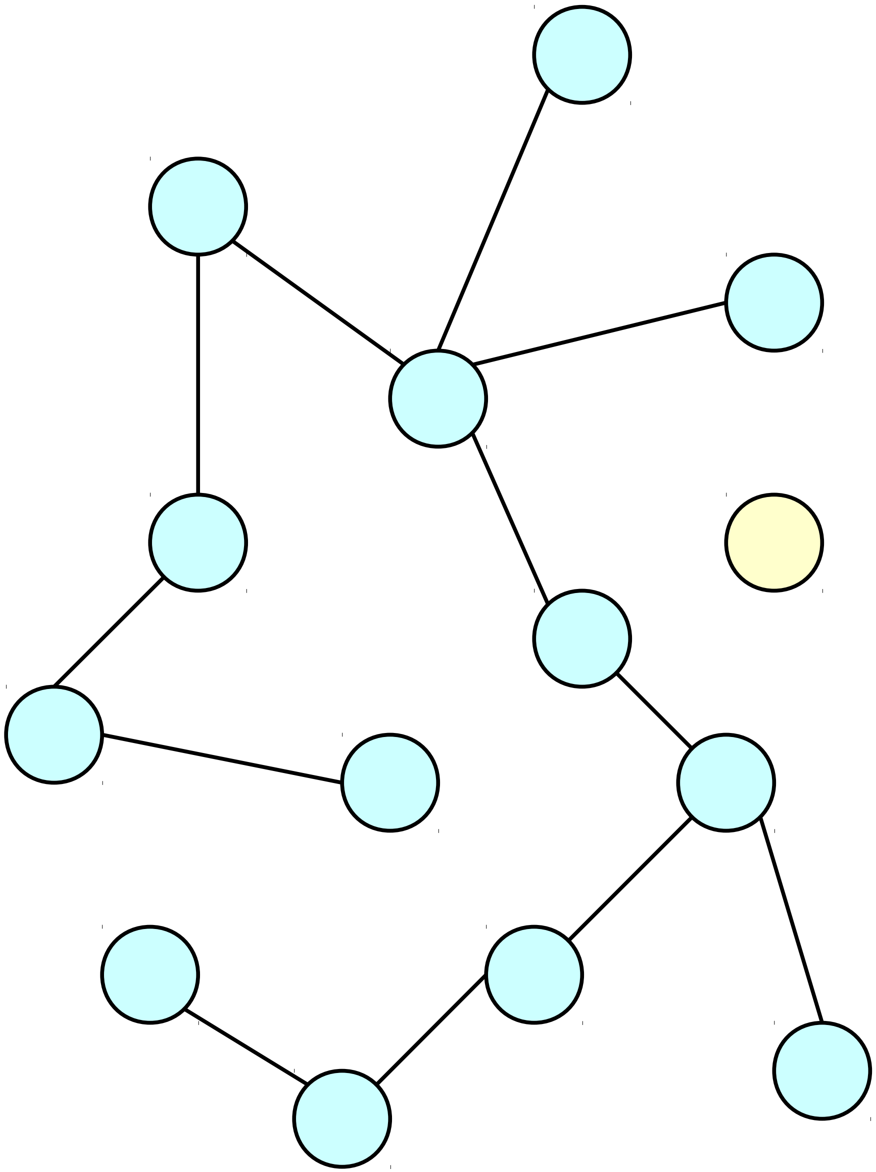
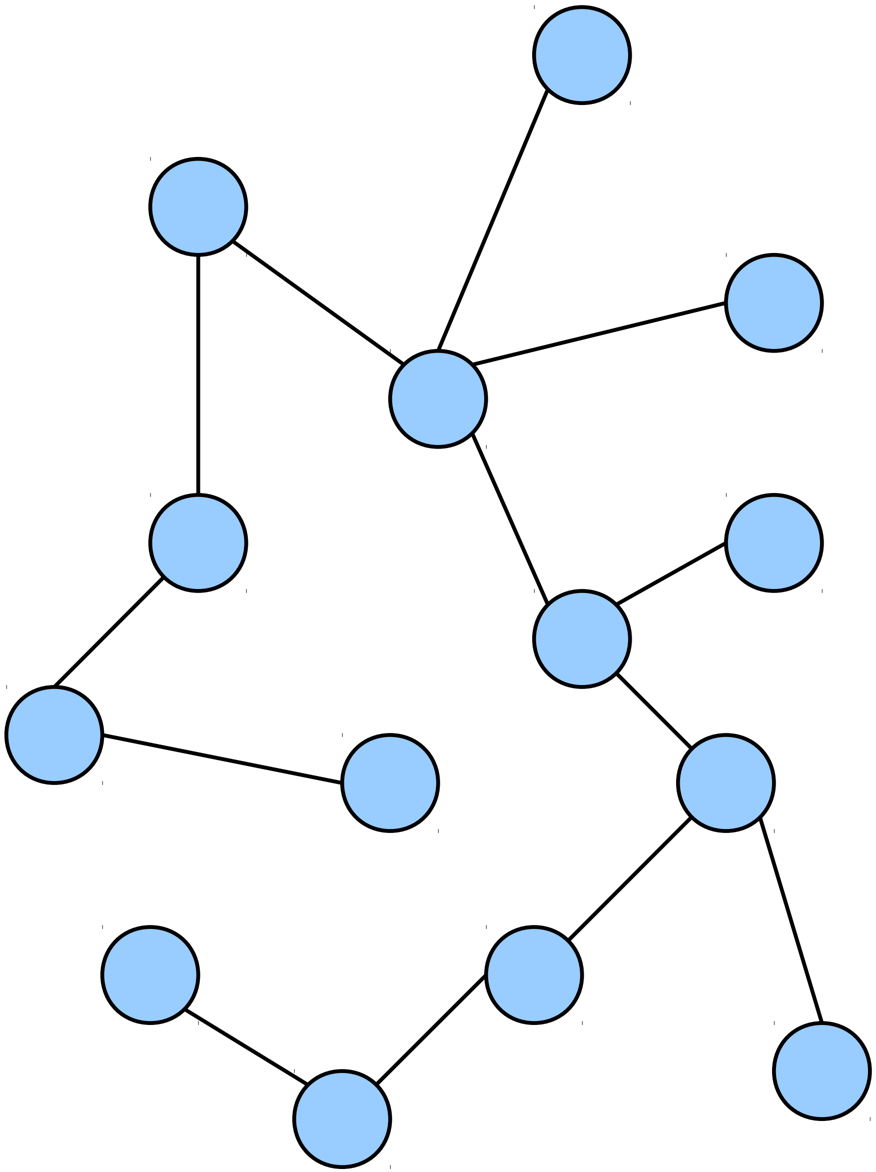


Trees

- ***Theorem:*** If T is a tree with $n \geq 1$ nodes, then T has exactly n-1 edges.
- ***Proof:*** Up next!



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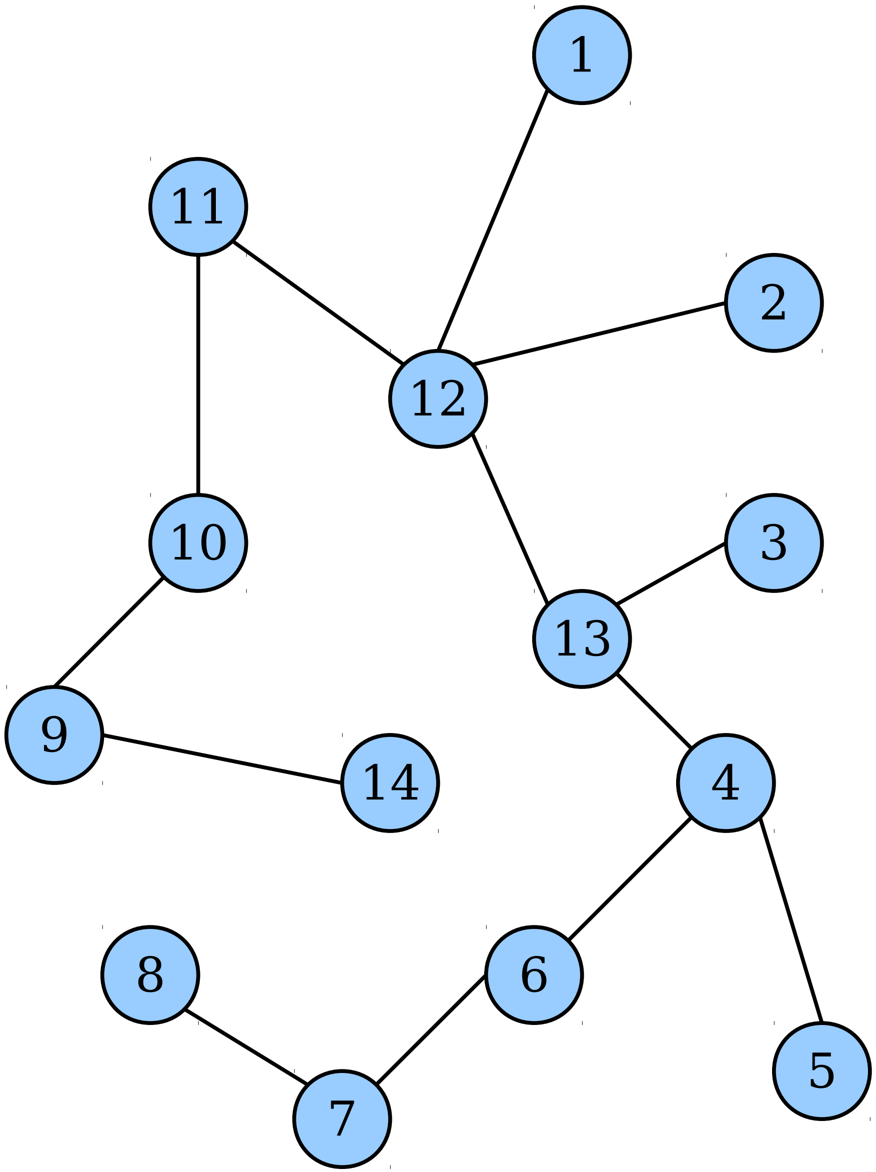

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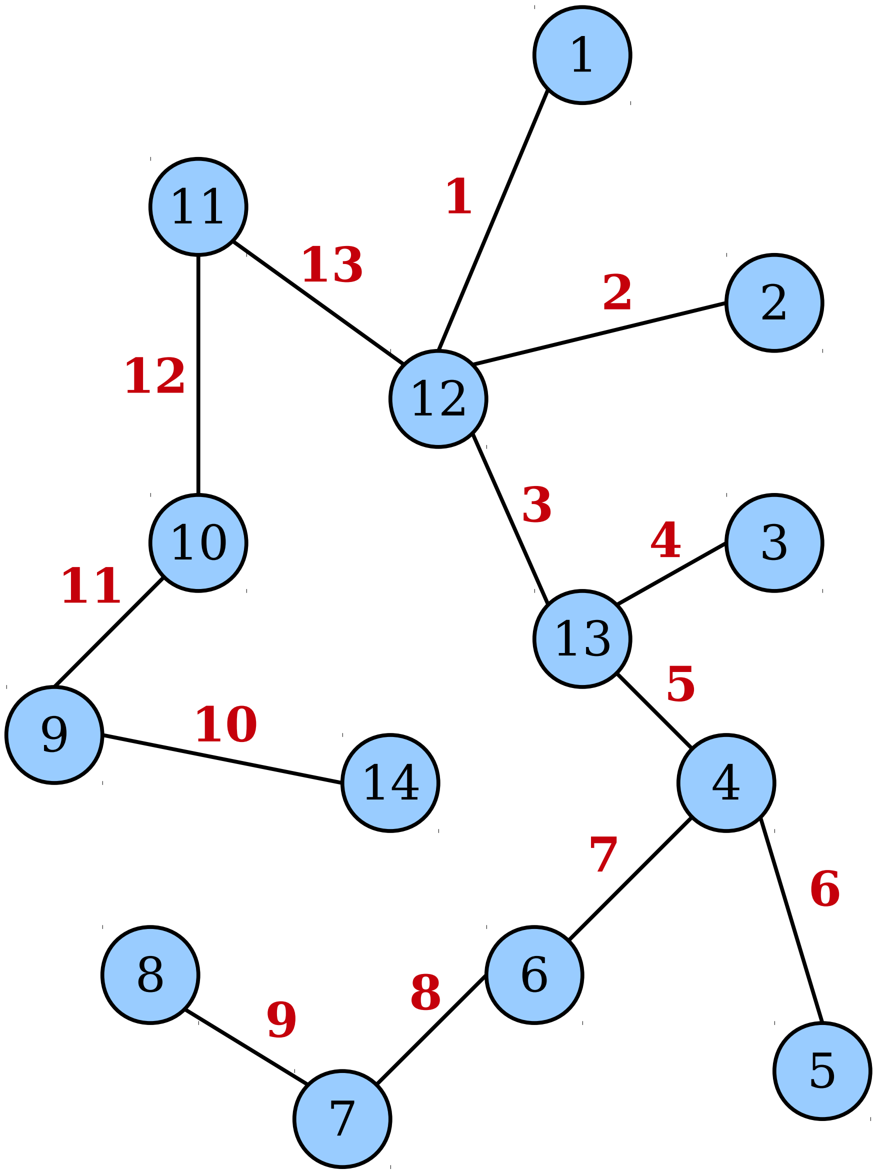

Our Base Case

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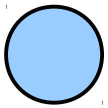

Assume any tree with at most *k* nodes has one more node than edge.

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Consider an arbitrary tree with *k*+1 nodes.

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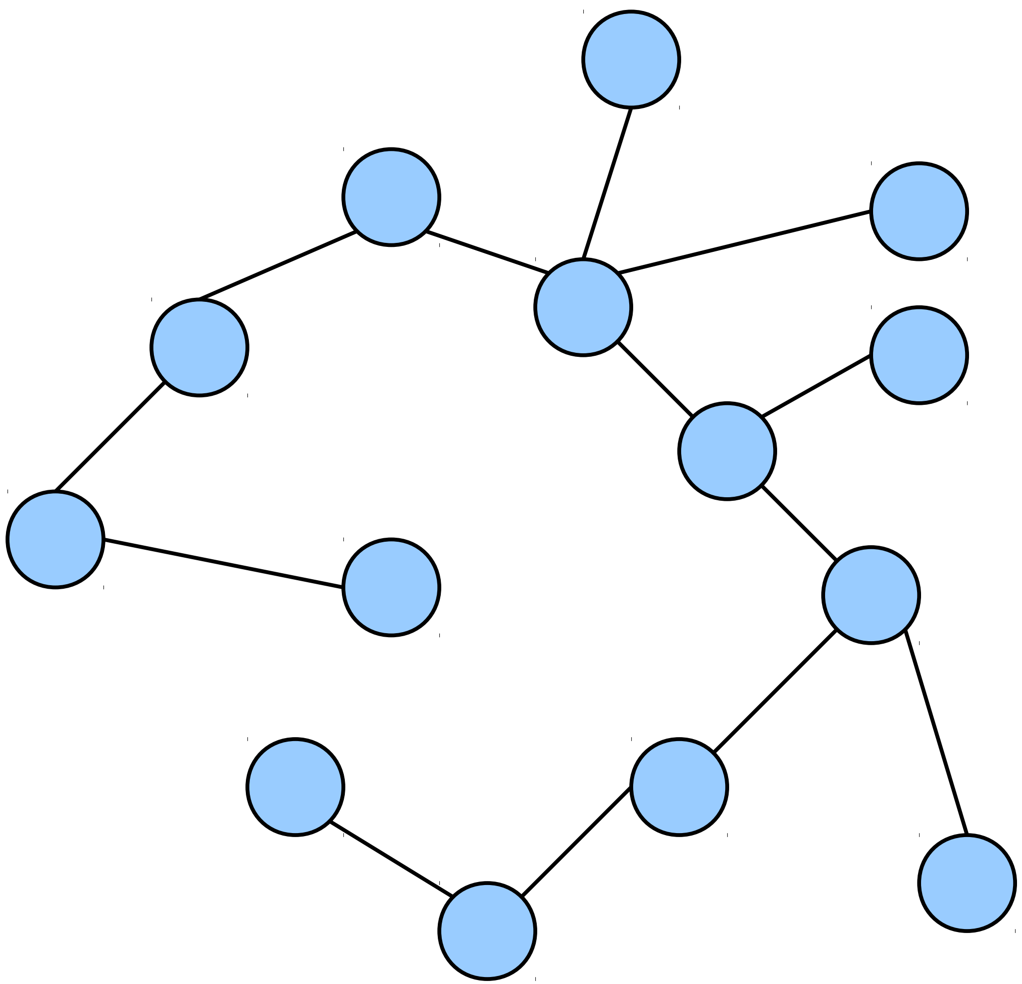

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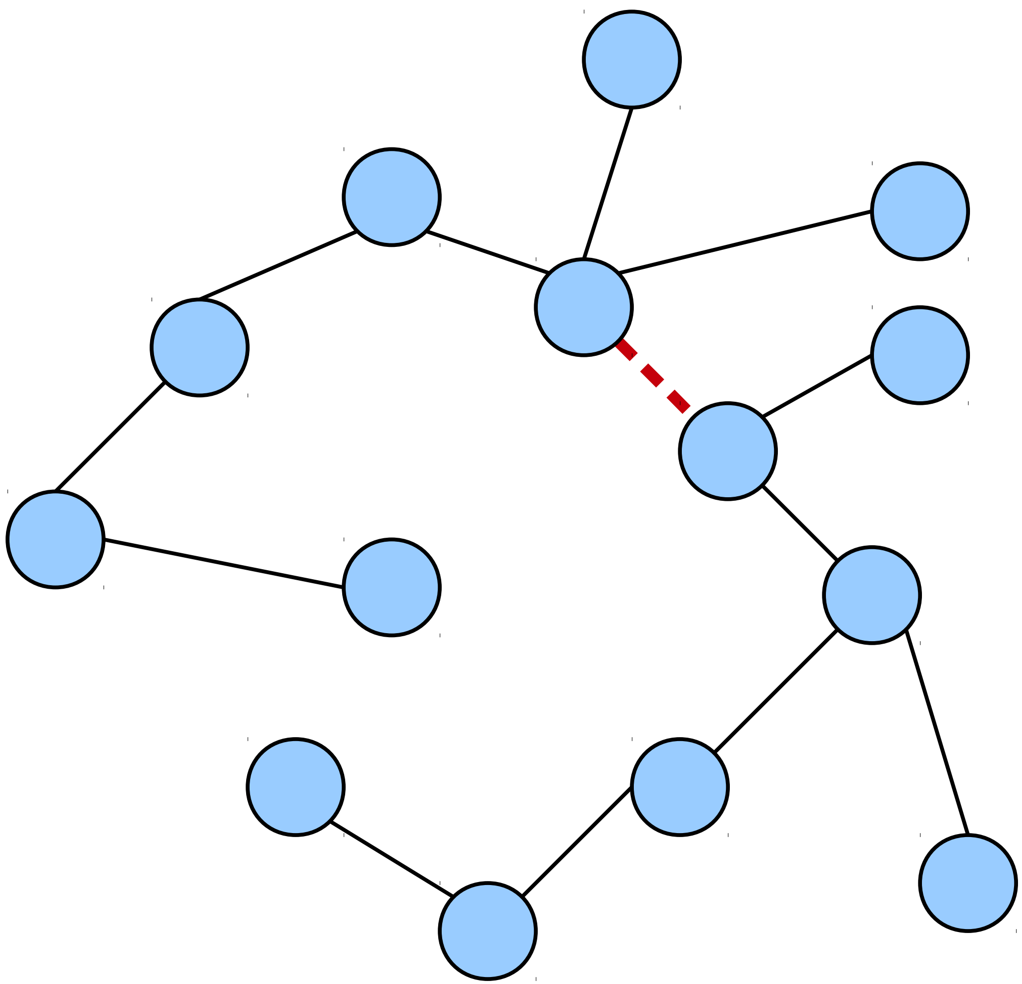

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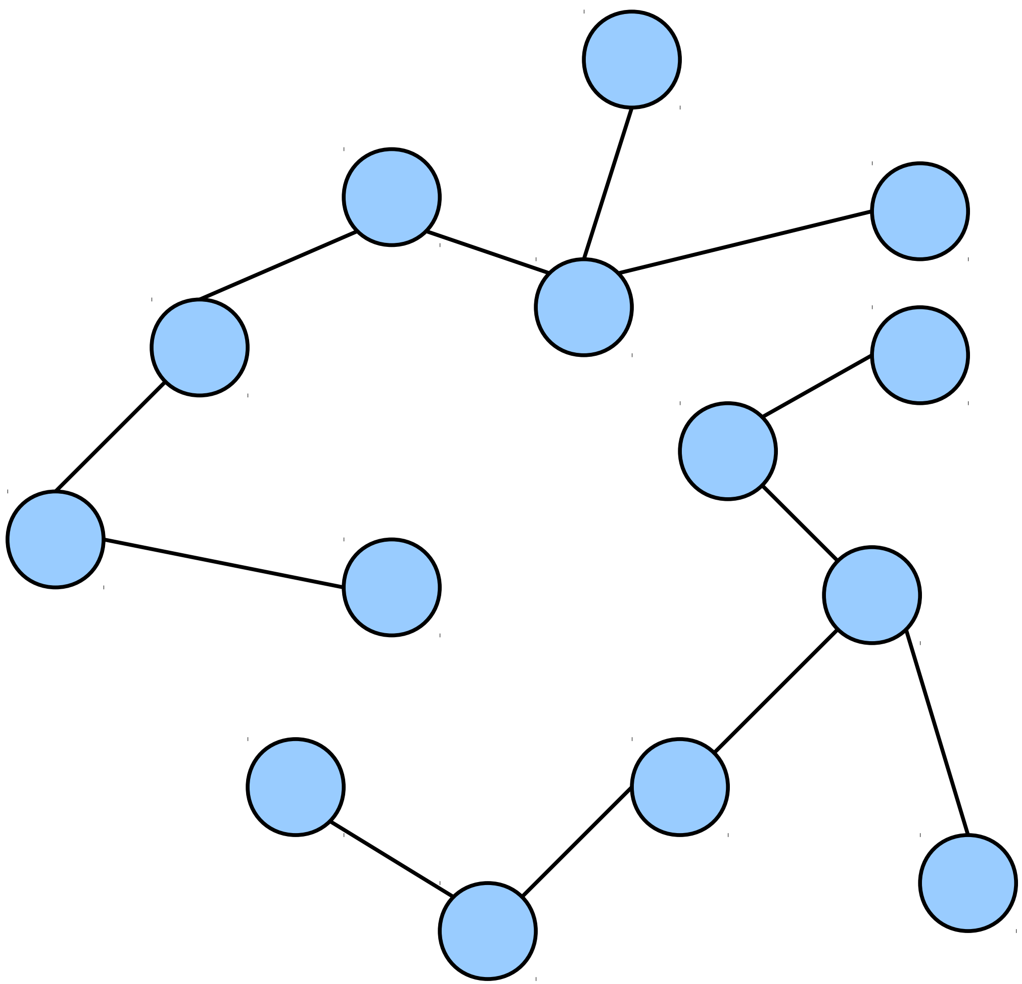

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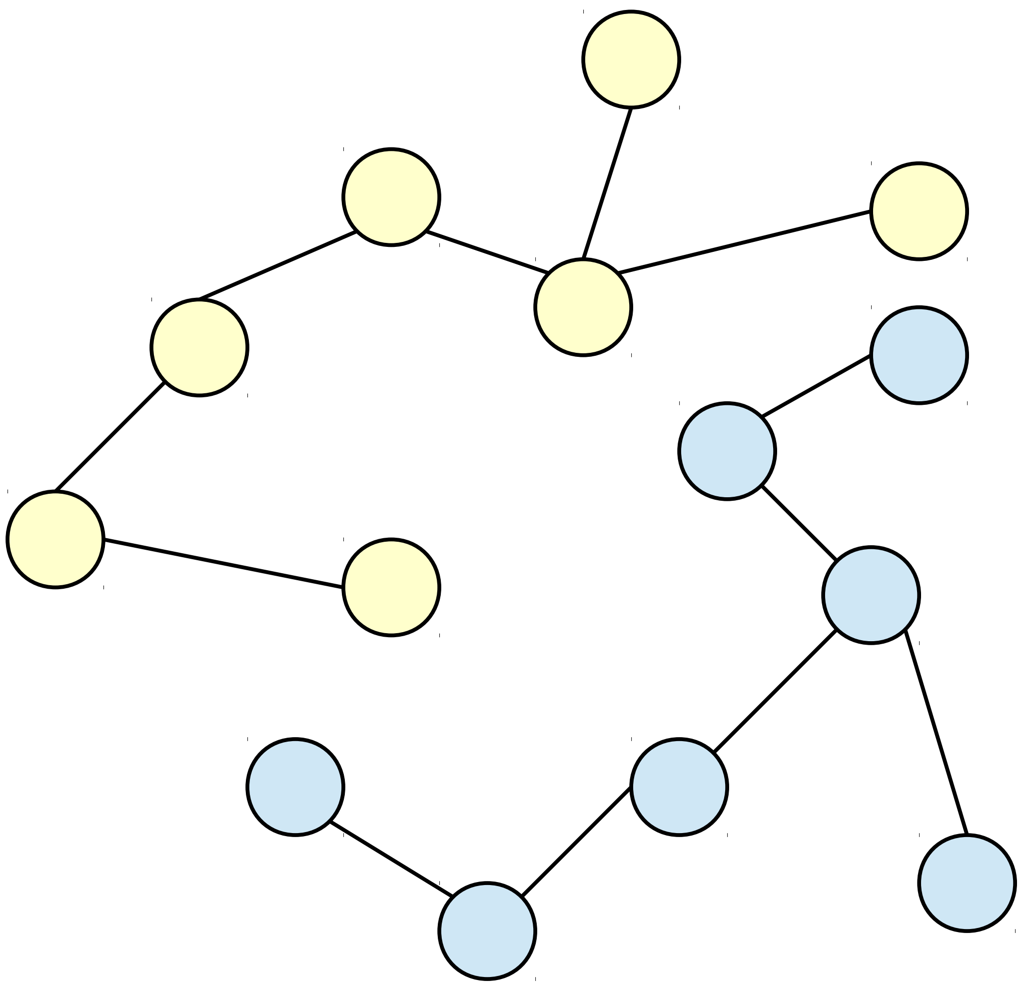

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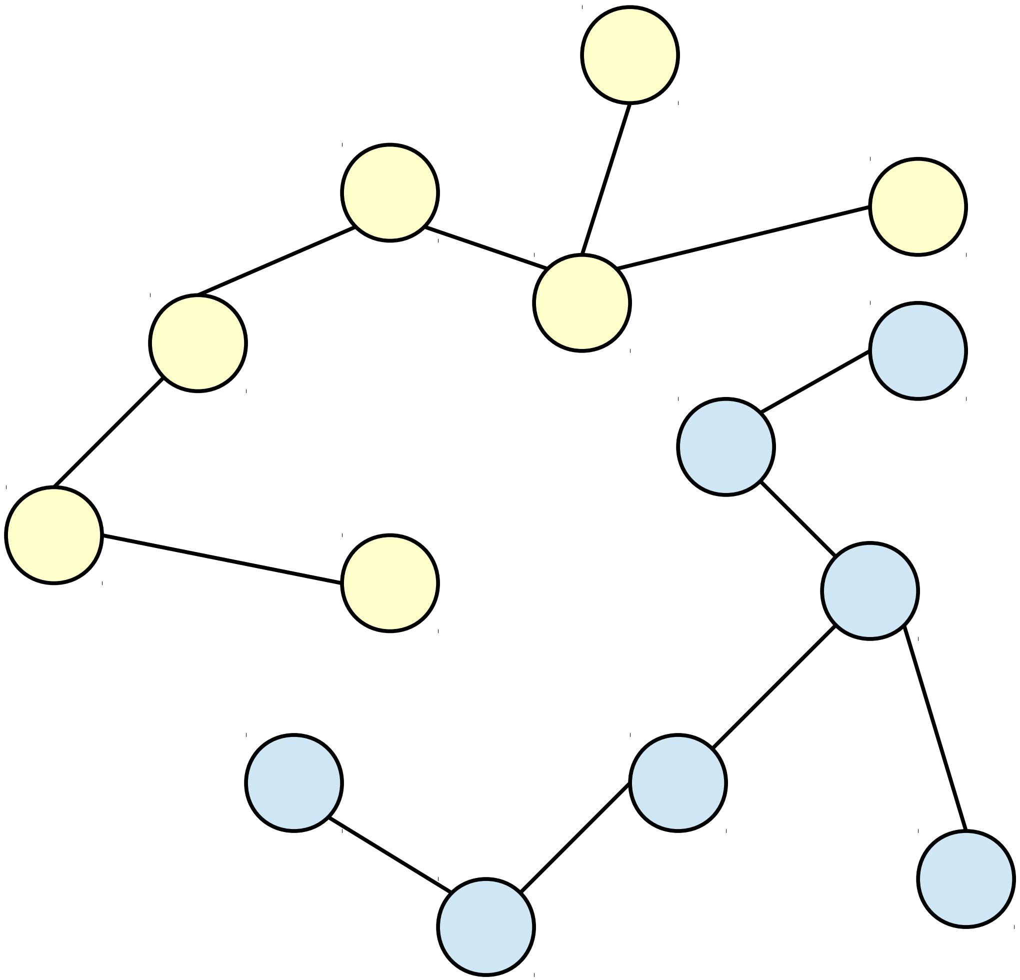
Consider an arbitrary tree with *k*+1 nodes.

Suppose there are r nodes in the yellow tree.



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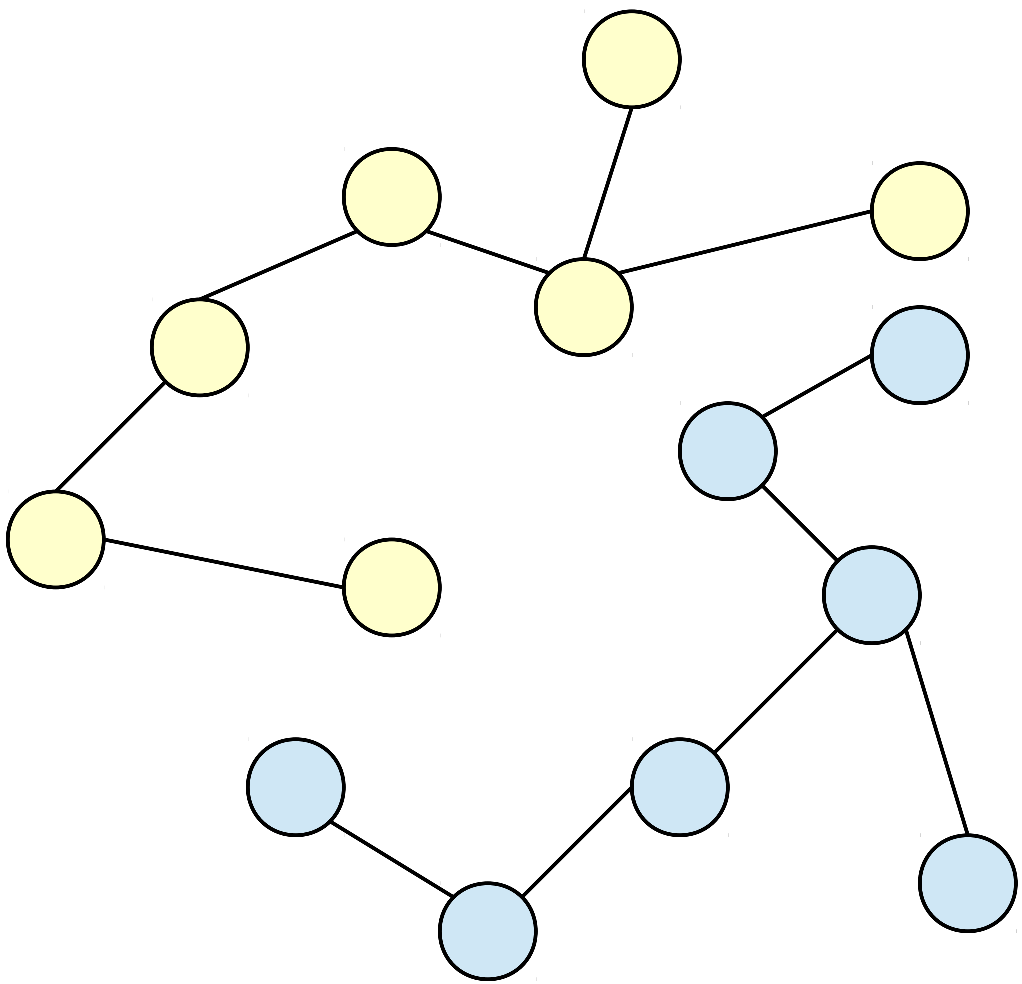
Consider an arbitrary tree with *k*+1 nodes.

Suppose there are r nodes in the yellow tree.

Then there are (*k*+1)-*r* nodes in the blue tree.



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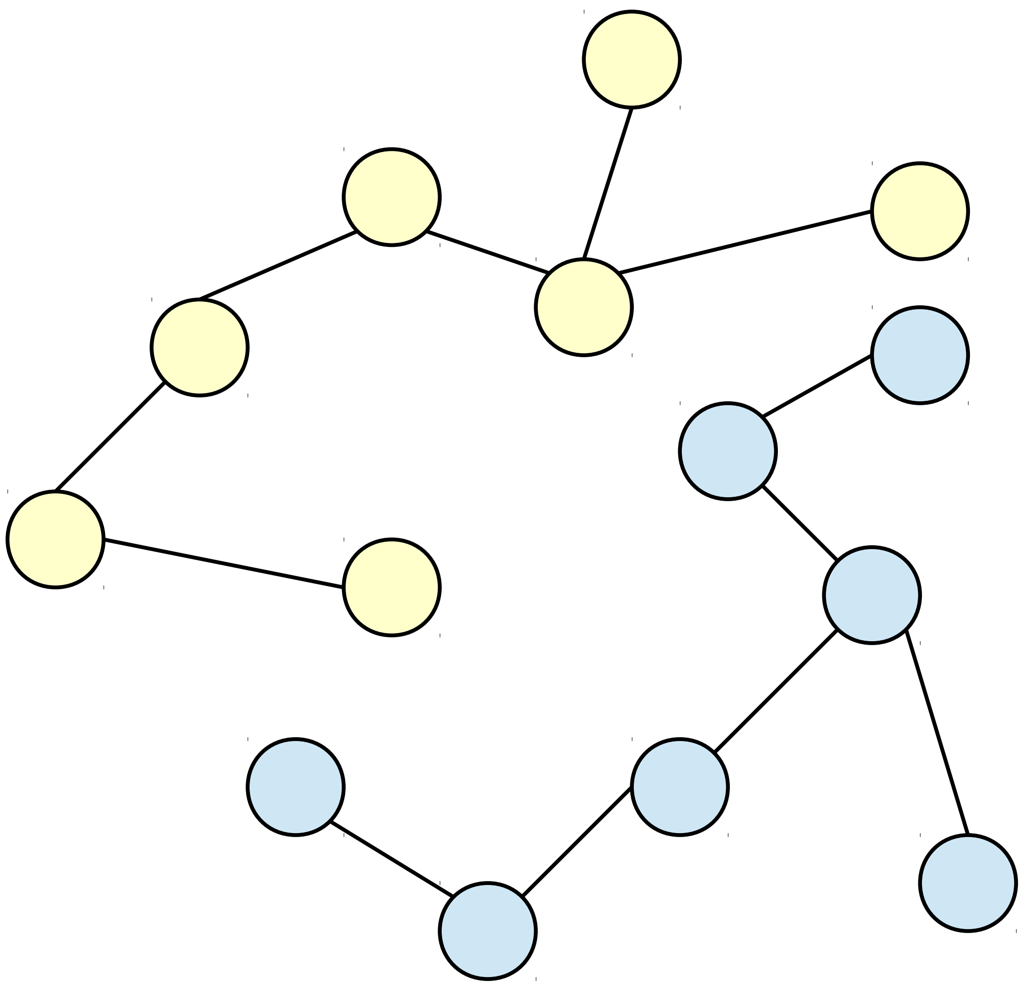

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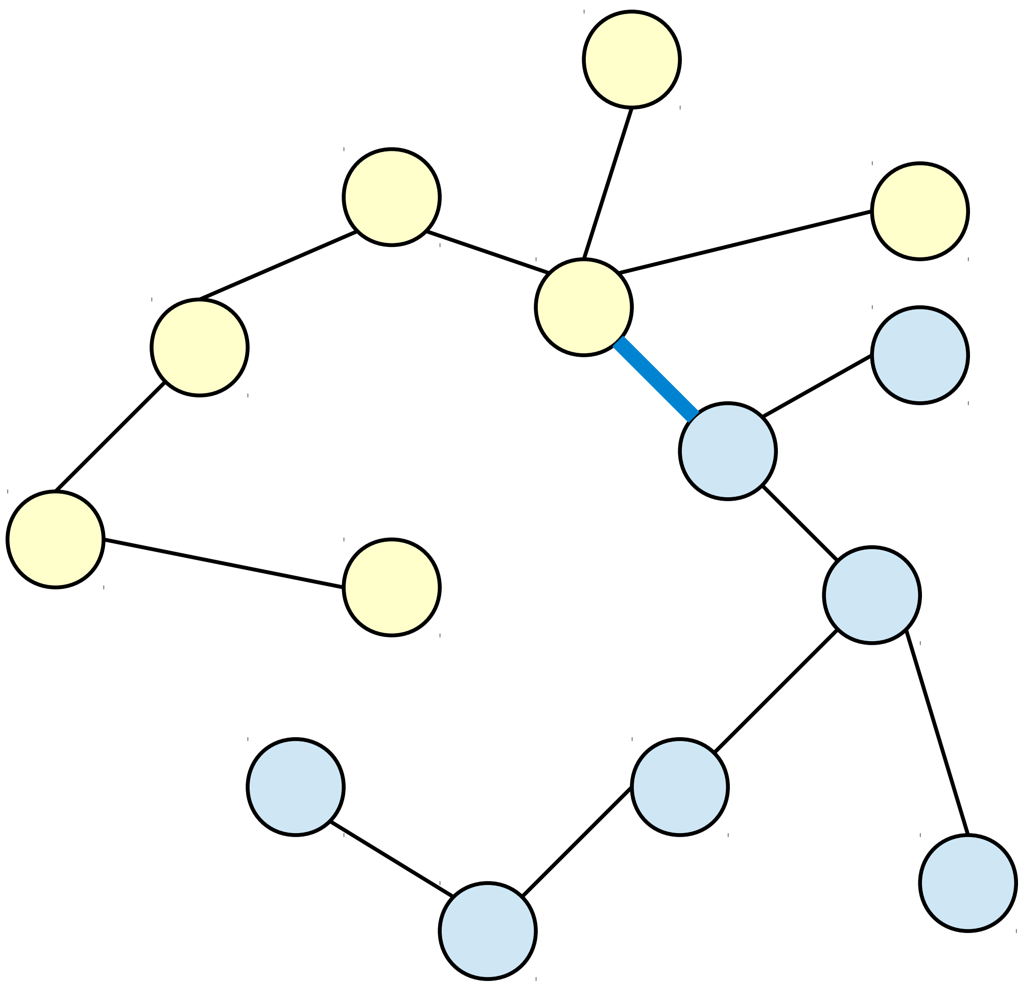

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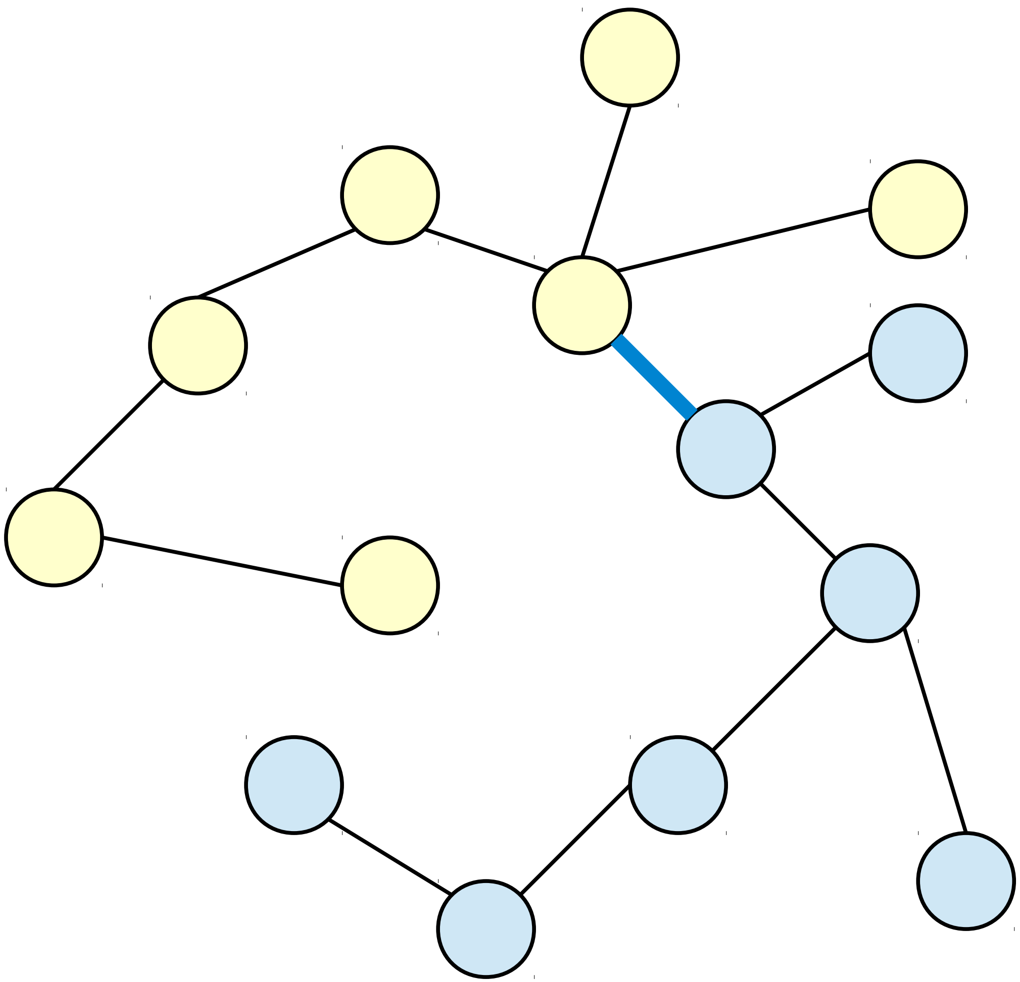

Consider an arbitrary tree with *k*+1 nodes.

Suppose there are r nodes in the yellow tree.

Then there are (k+1)-r nodes in the blue tree.

There are r-1 edges in the yellow tree and k-r edges in the blue tree.

Adding in the initial edge we cut, there are
r-1 + k-r + 1 = k edges in the original tree.

Theorem: If T is a tree with $n \geq 1$ nodes, then T has n-1 edges.

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Which of the following best describes the structure of the inductive step in this proof?

A. Assume $P(1)$, then prove $P(k+1)$.
B. Assume $P(k)$, then prove $P(k+1)$.
C. Assume $P(1)$, then prove $P(1)$, …, $P(k)$, and $P(k+1)$.
D. Assume $P(1)$, …, and $P(k)$, then prove $P(k+1)$.
E. None of these, or more than one of these.

Therefore, T_1 and T_2 each have between 1 and k nodes. We can then apply our inductive hypothesis to see that T_1 has r-1 edges and T_2 has k-r edg … $r) = k$, as req … n. ■

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then ***A***, …, or ***E***.

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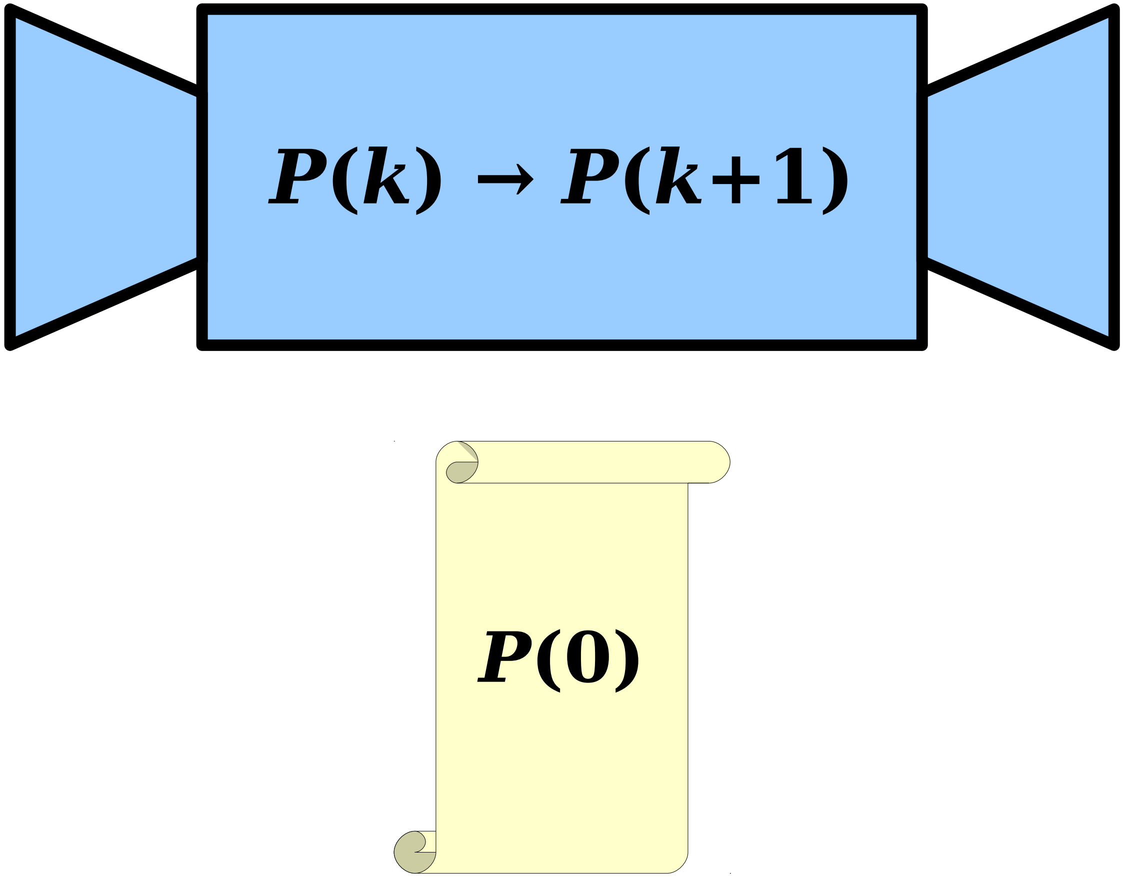
Complete Induction

- If the following are true:
  - $P(0)$ is true, and
  - If $P(0), P(1), P(2), \ldots, P(k)$ are true, then $P(k+1)$ is true as well.

  then $P(n)$ is true for all $n \in \mathbb{N}$.
- This is called the ***principle of complete induction*** or the ***principle of strong induction***.
  - (This also works starting from a number other than 0; just modify what you're assuming appropriately.)

Review: Induction as a Machine



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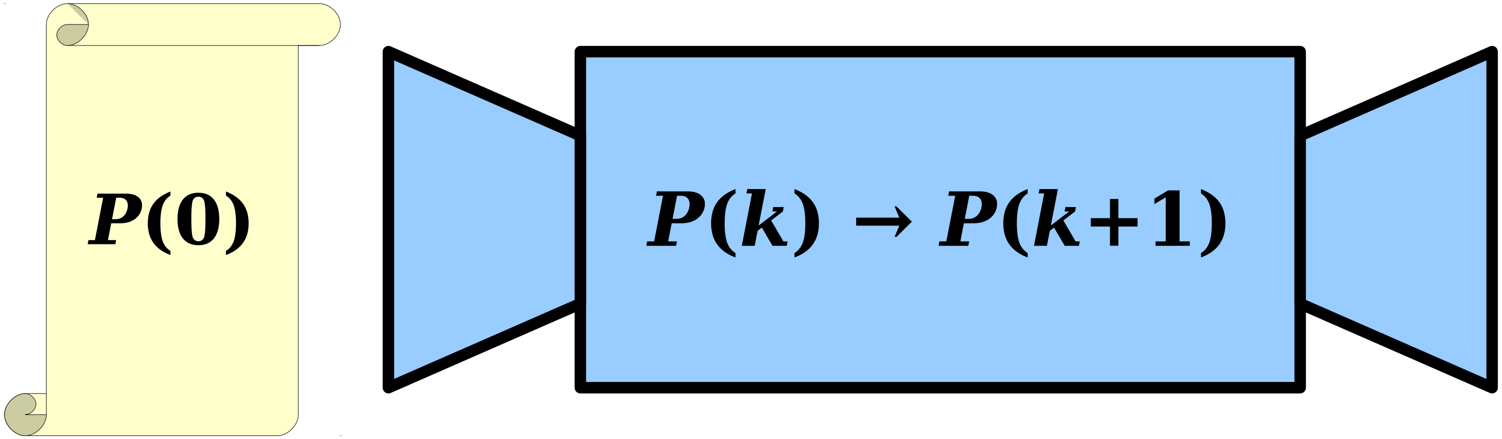

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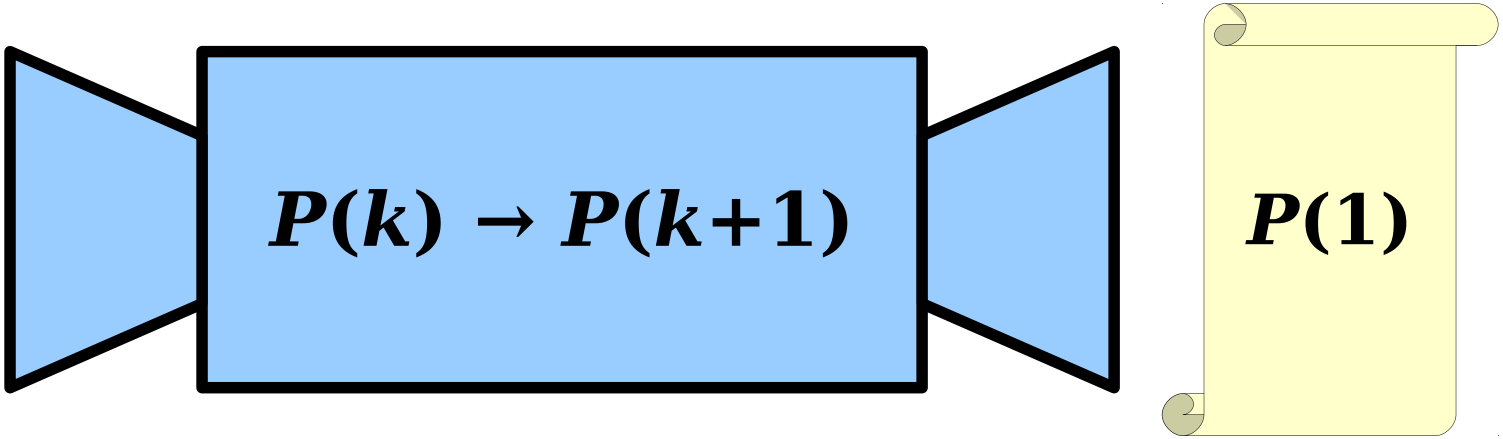

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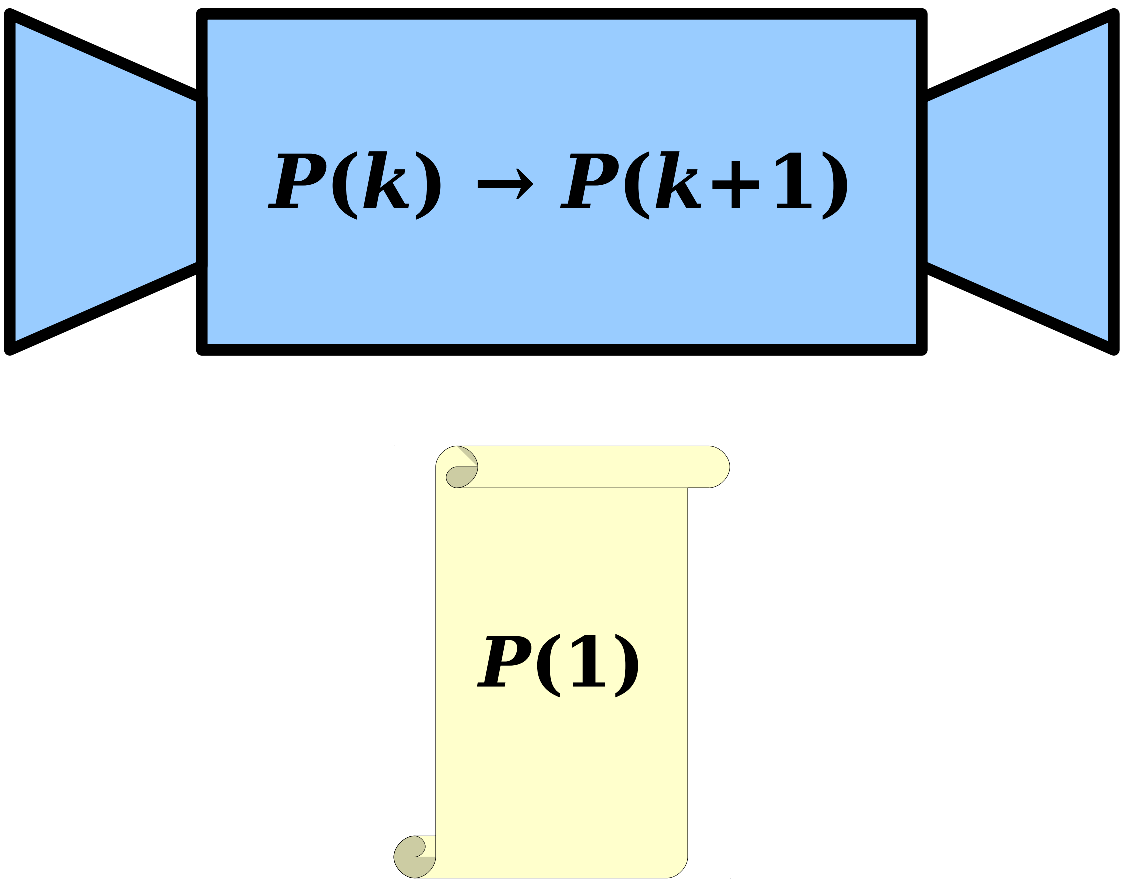

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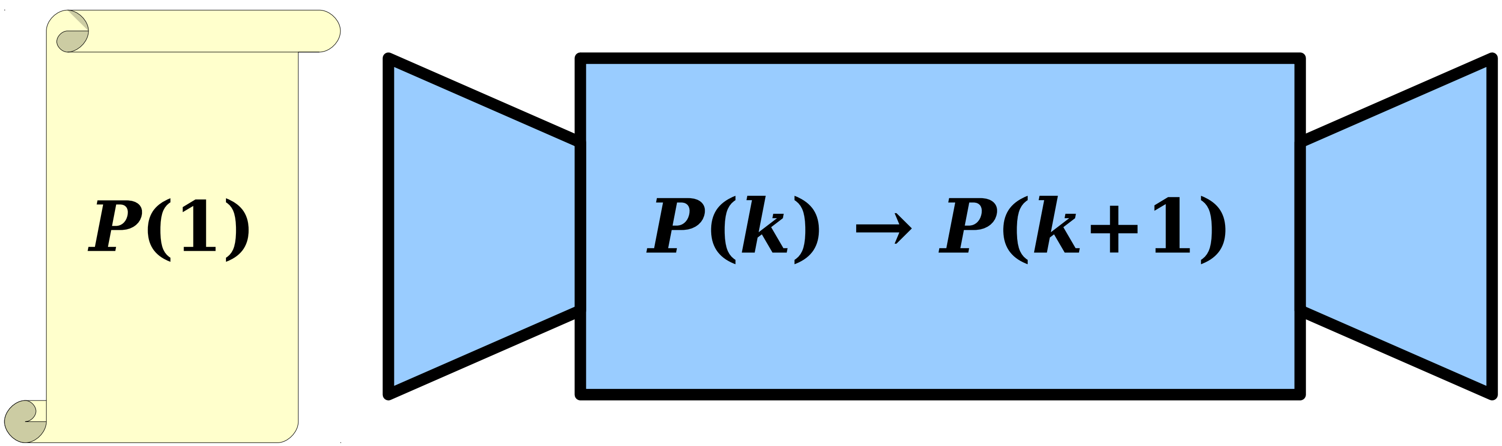

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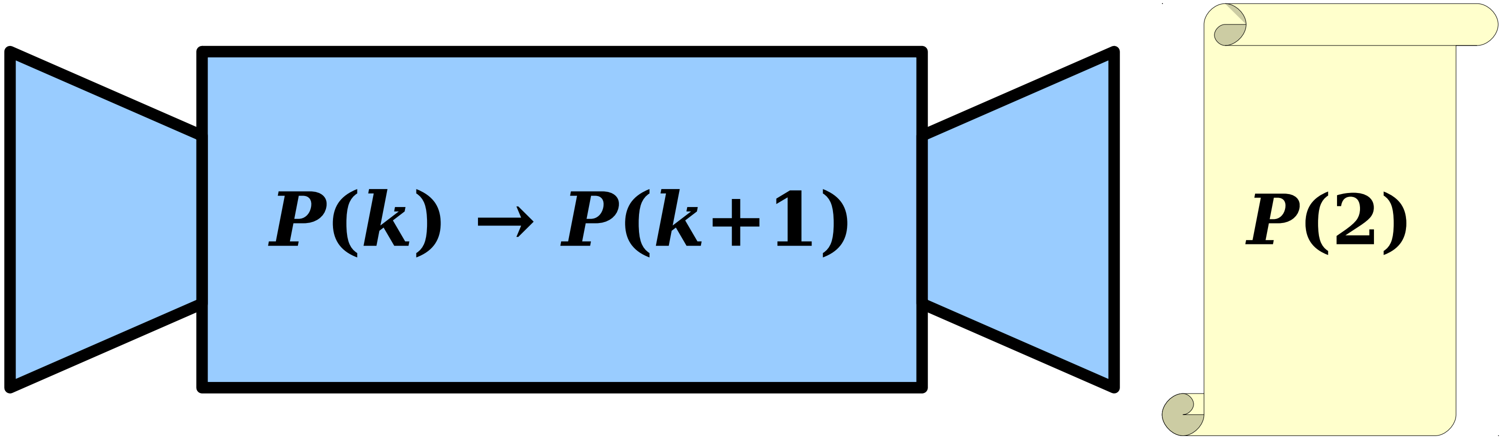

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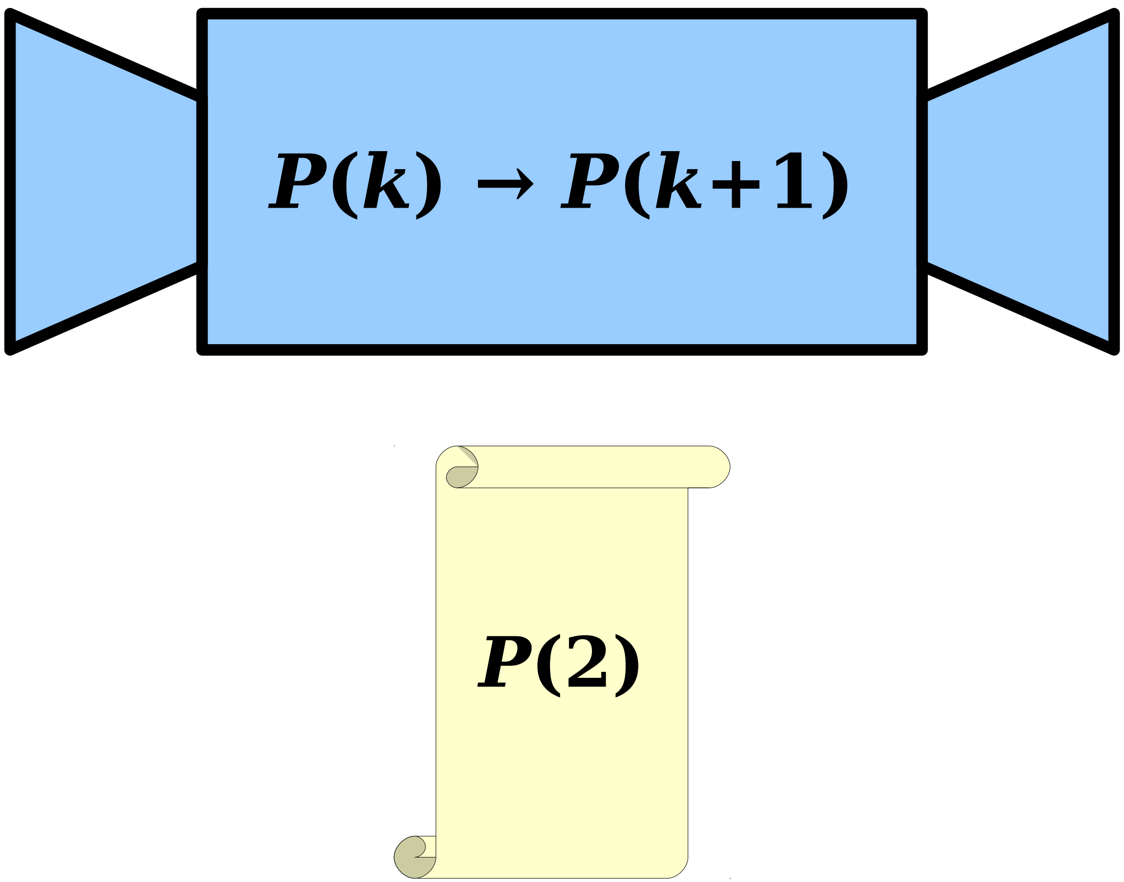

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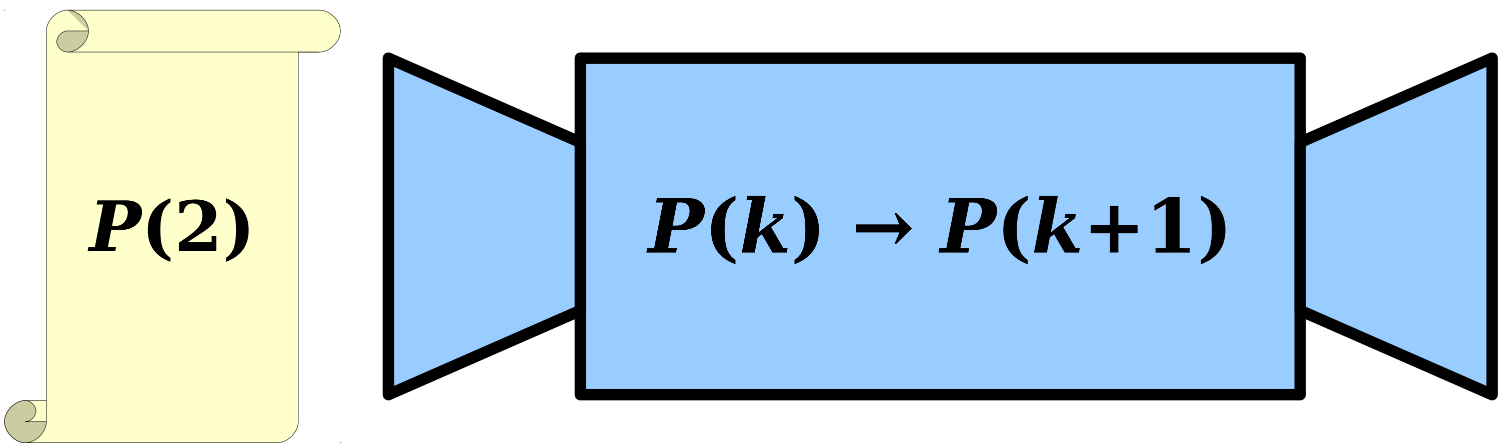

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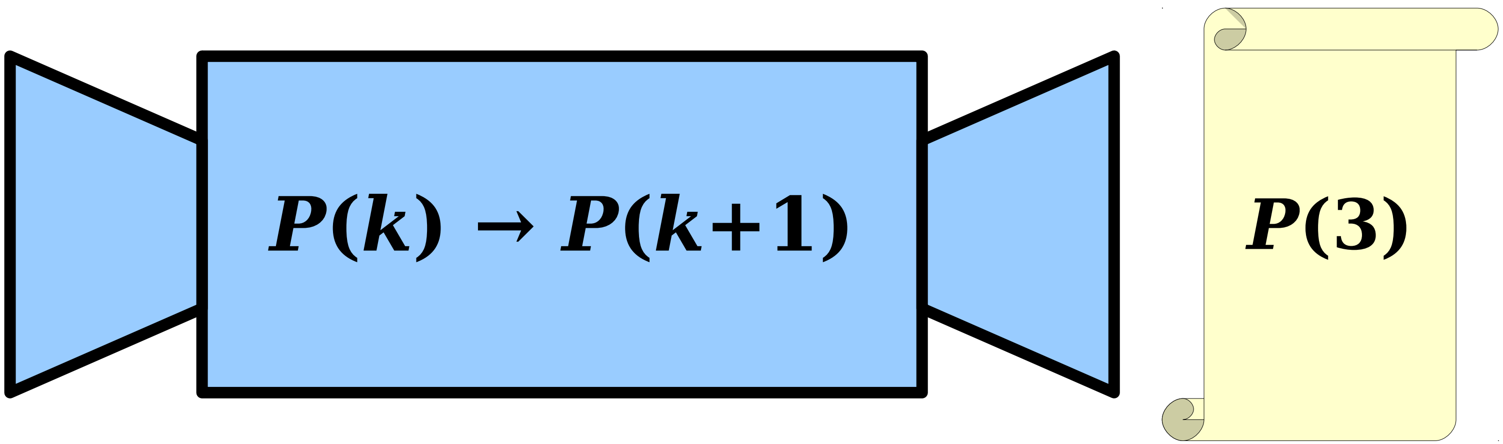

An Observation

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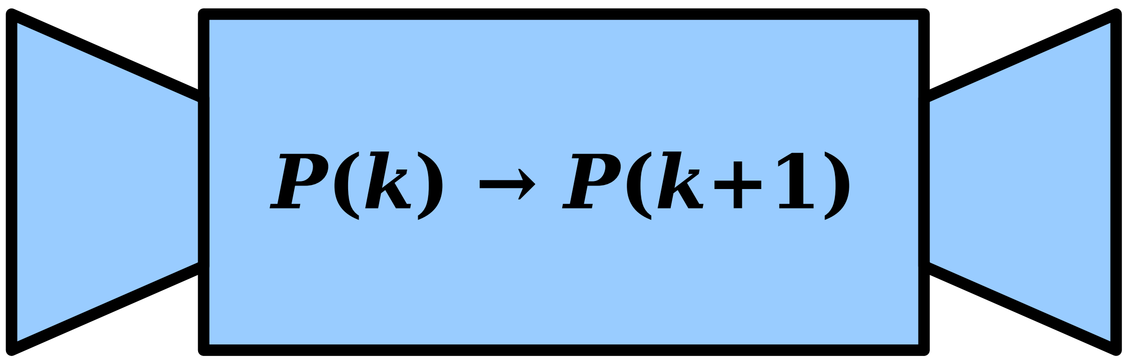

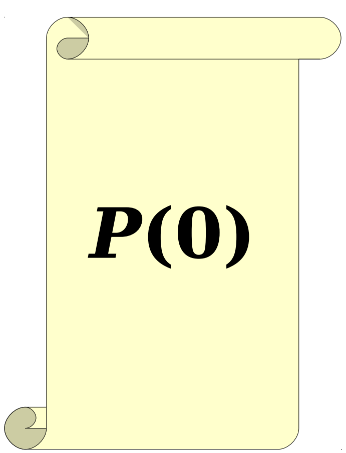

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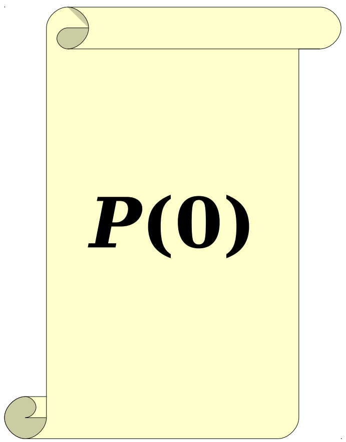

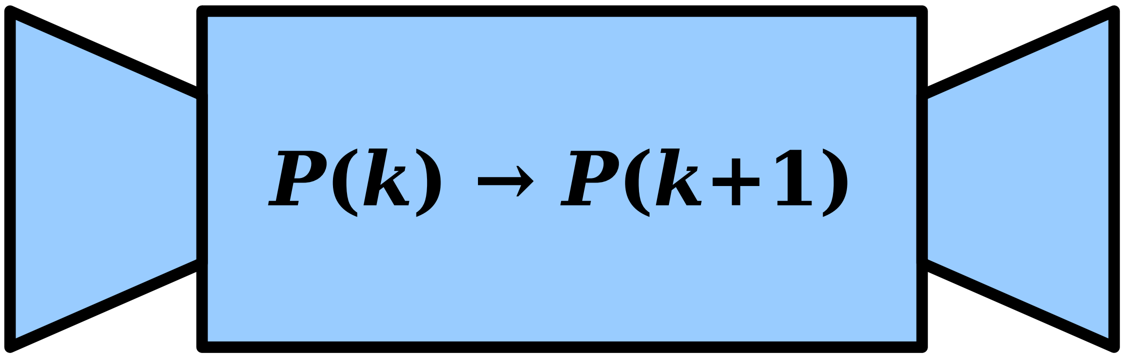

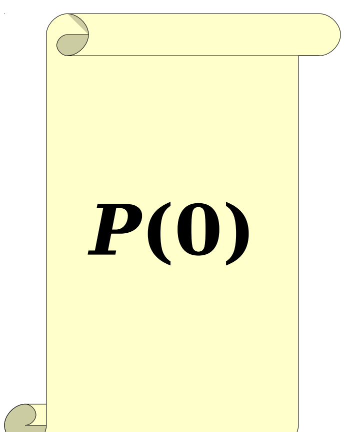

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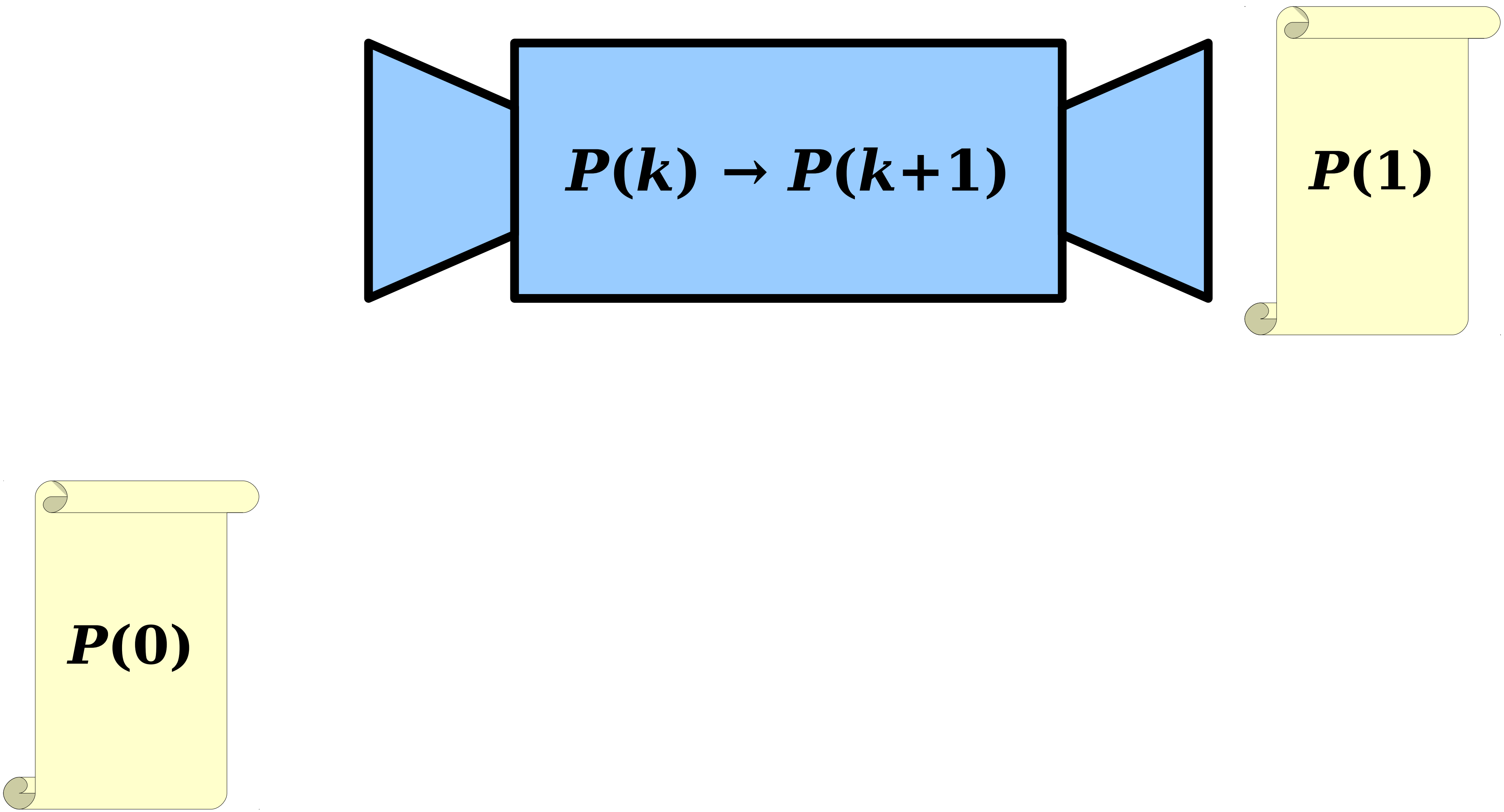

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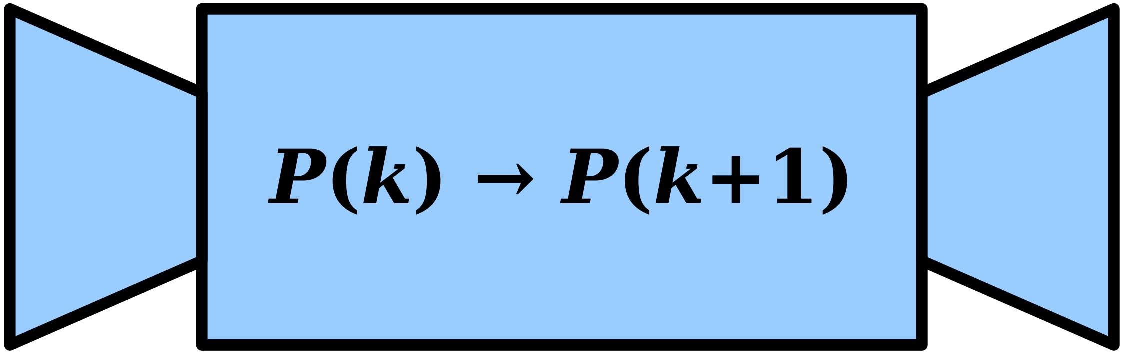

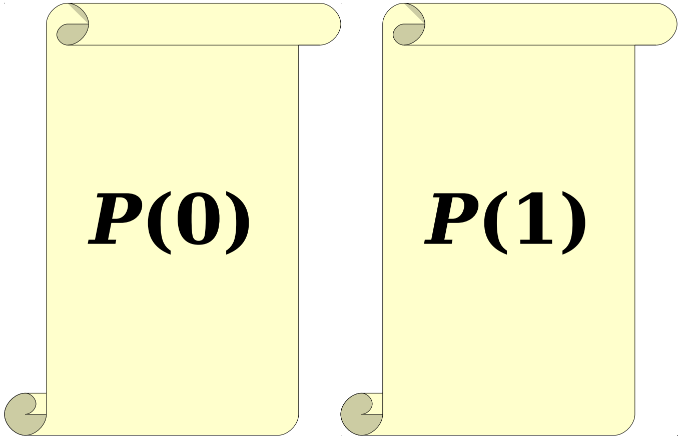

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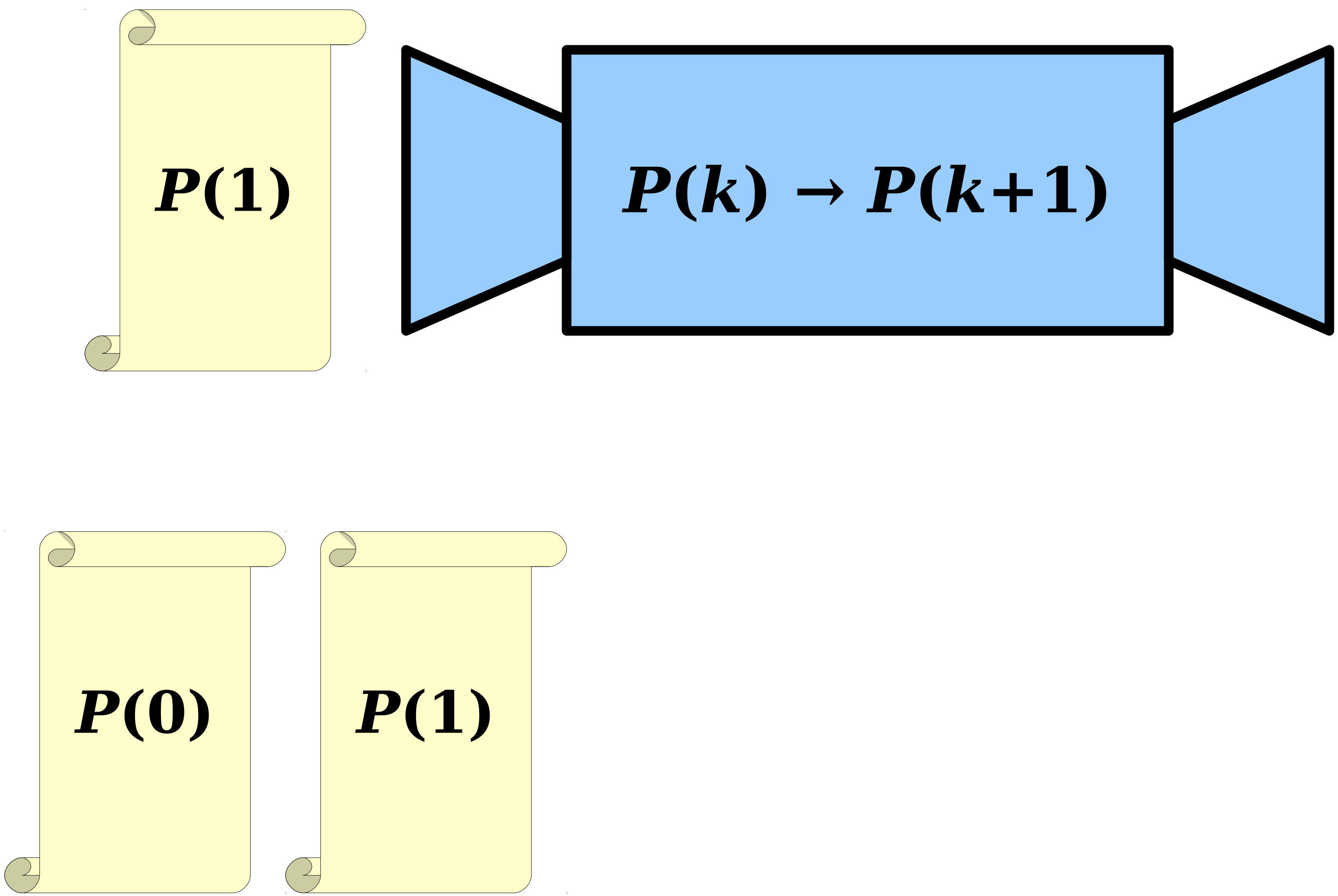

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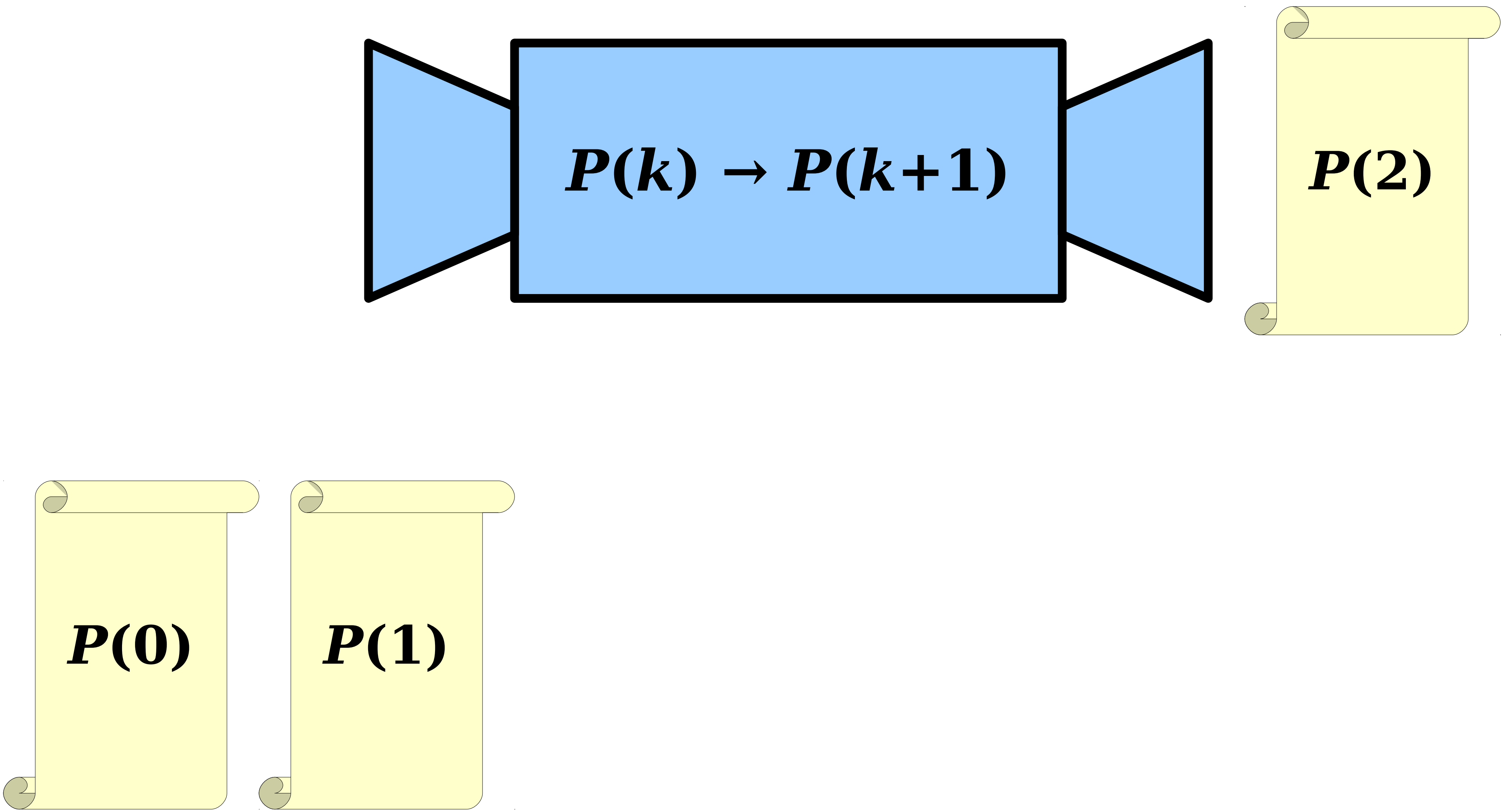

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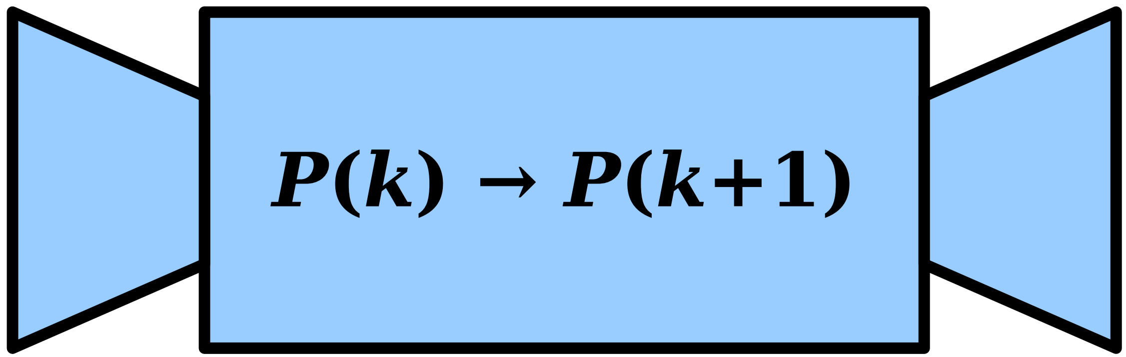

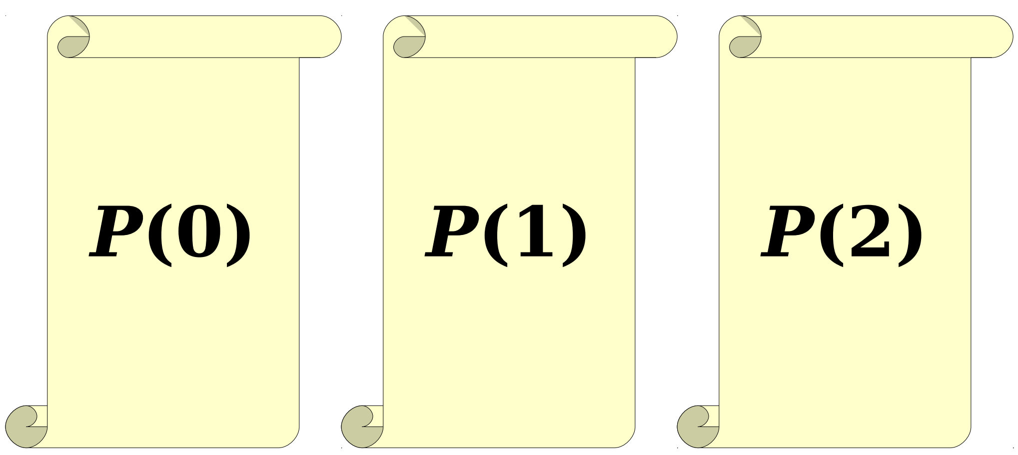

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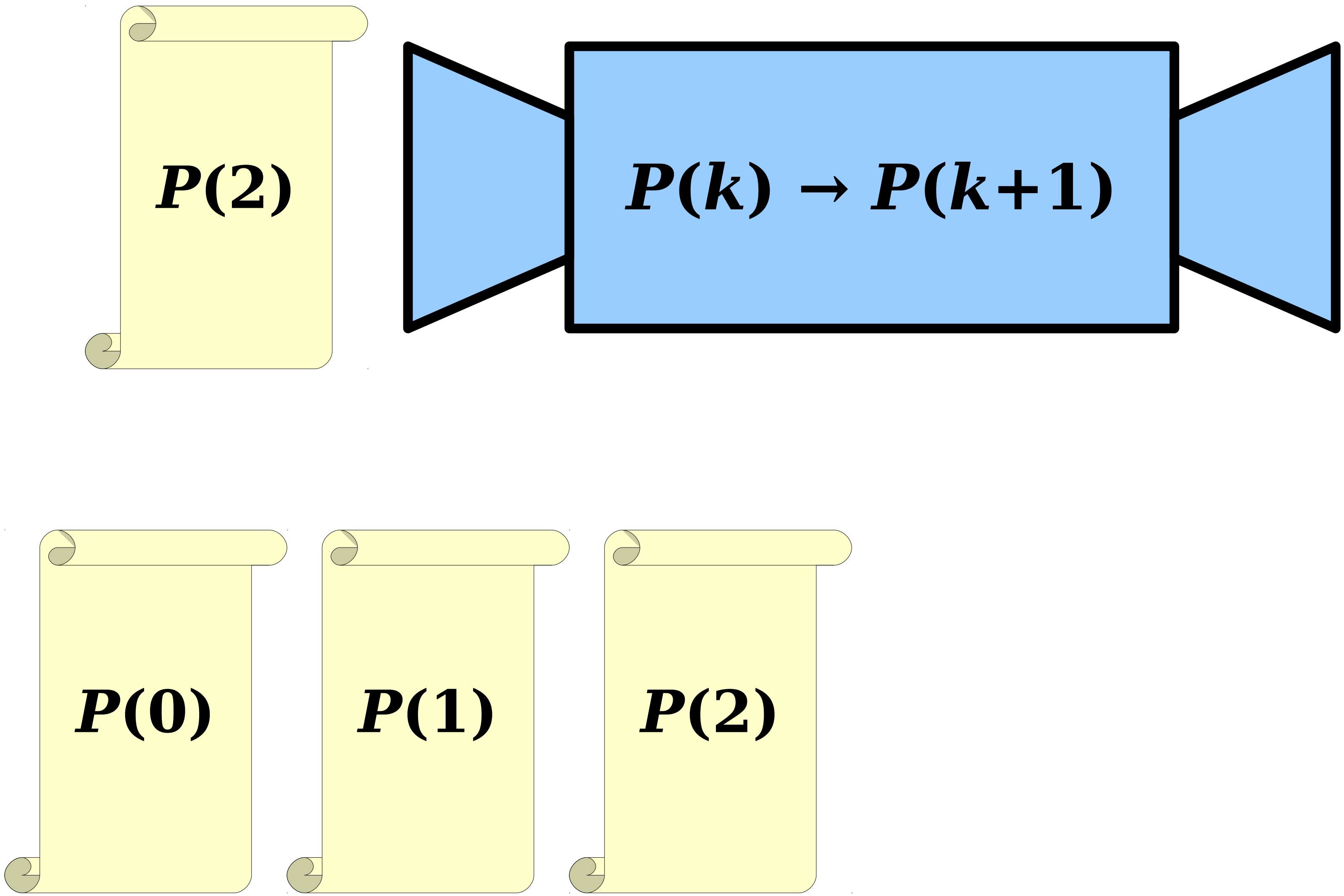

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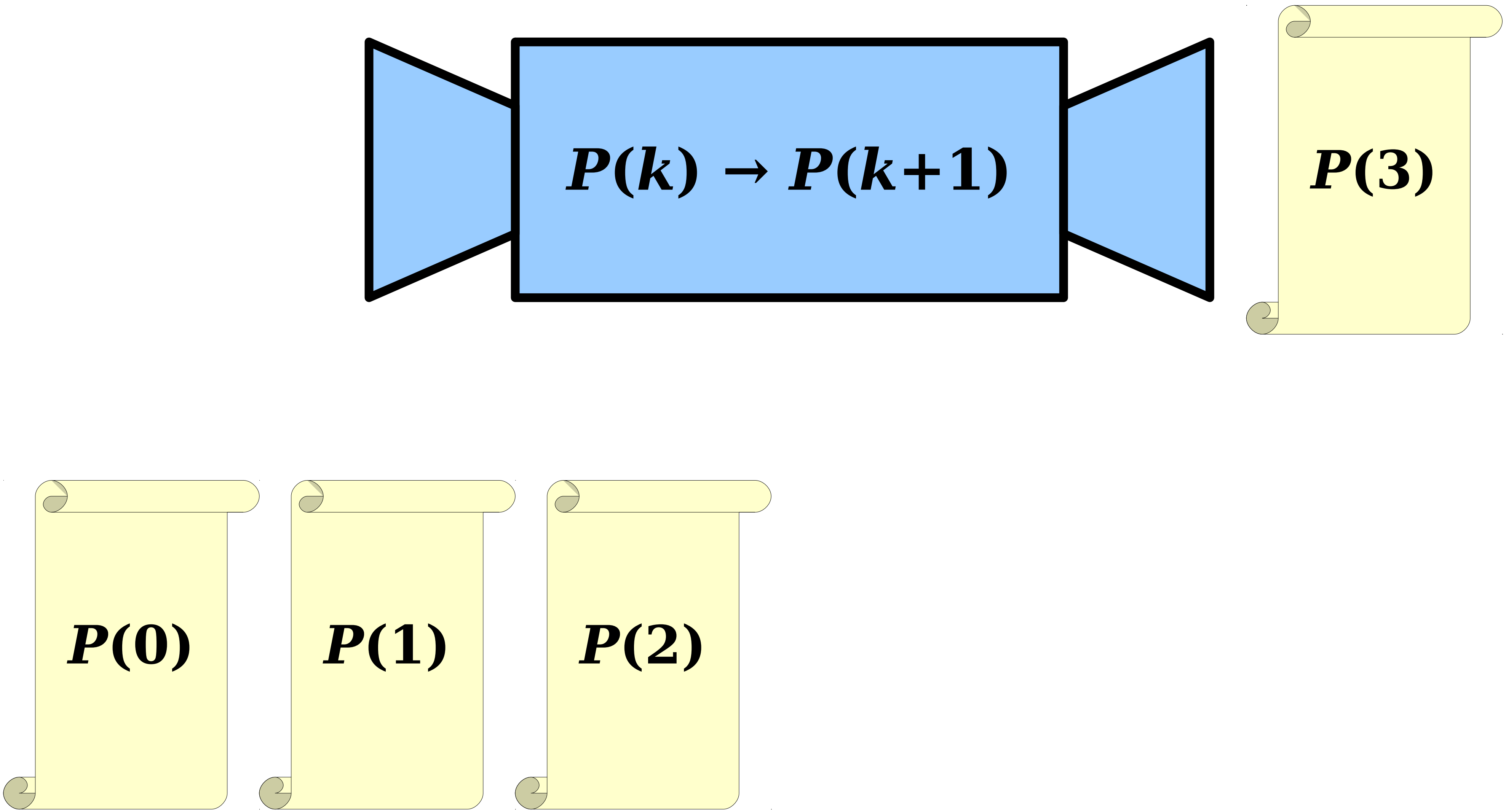

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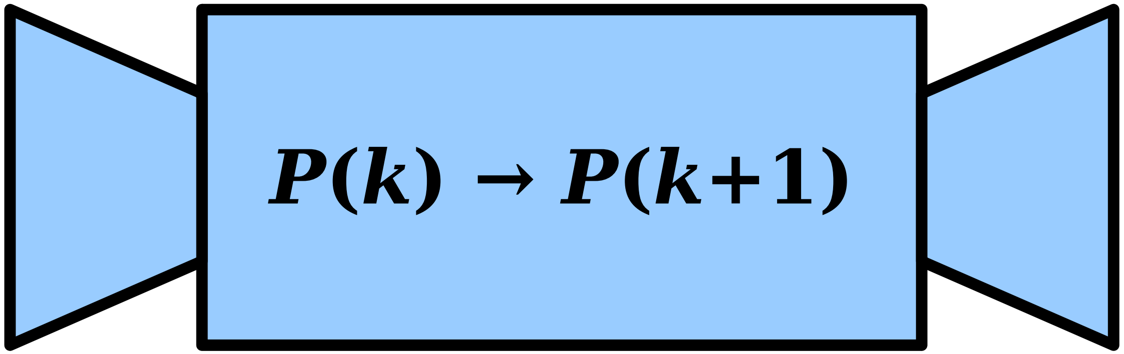

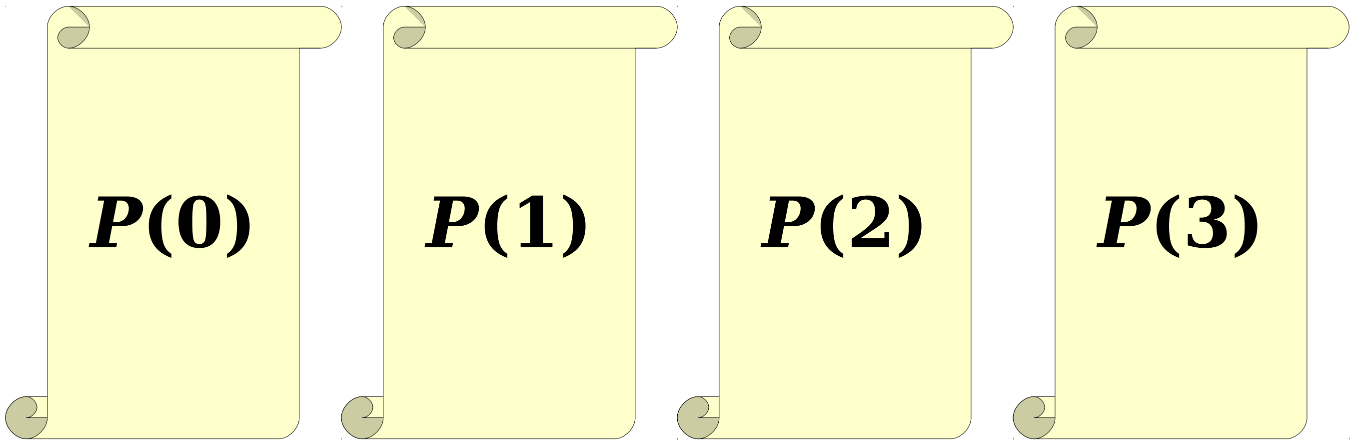

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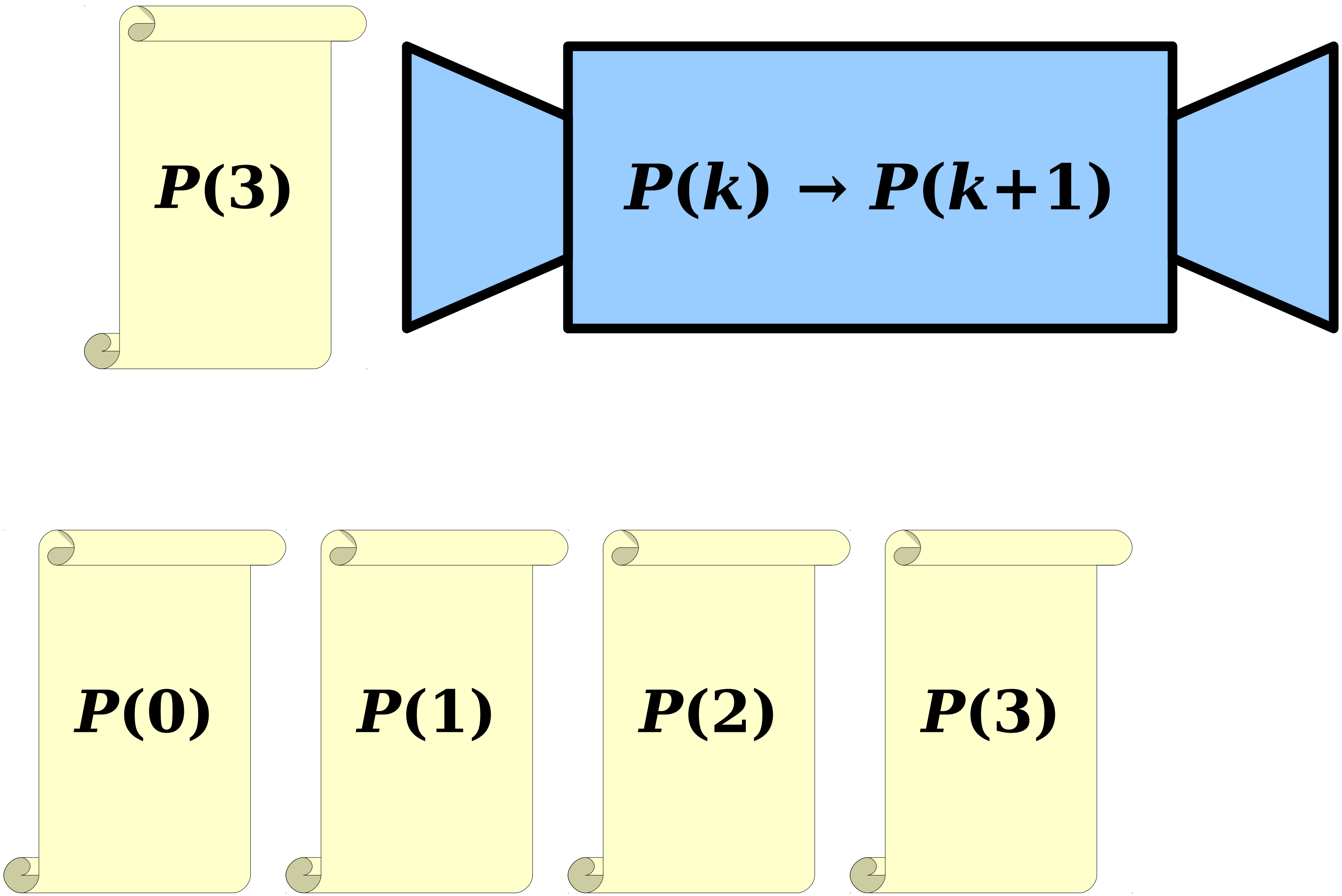

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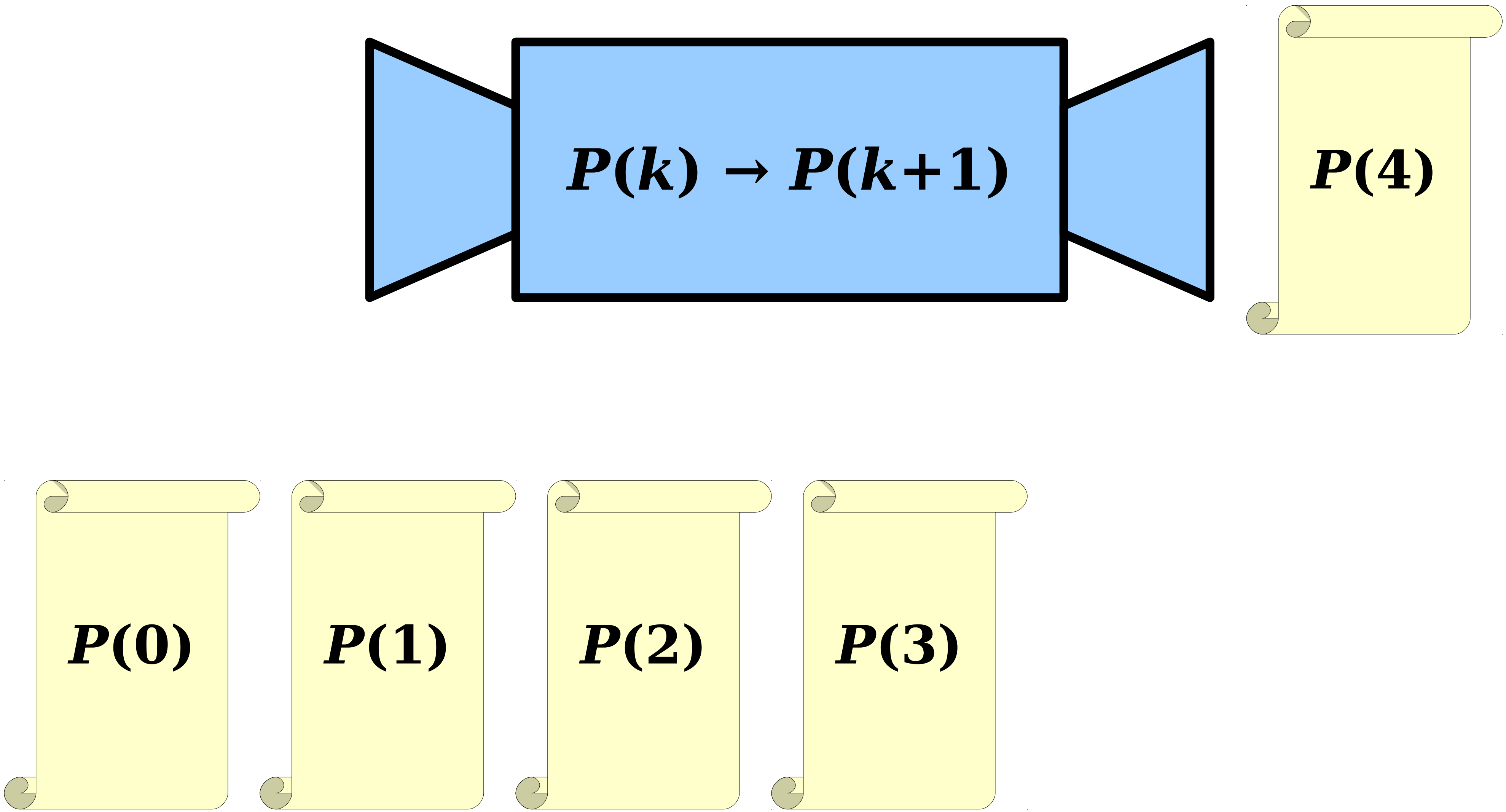

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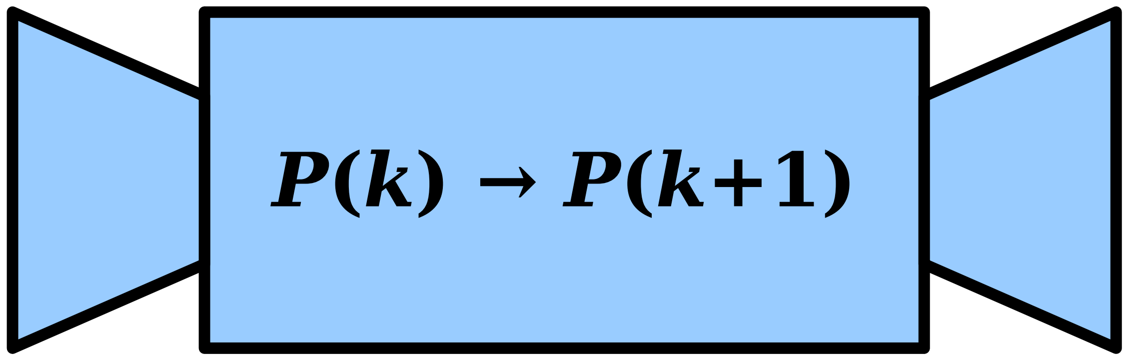

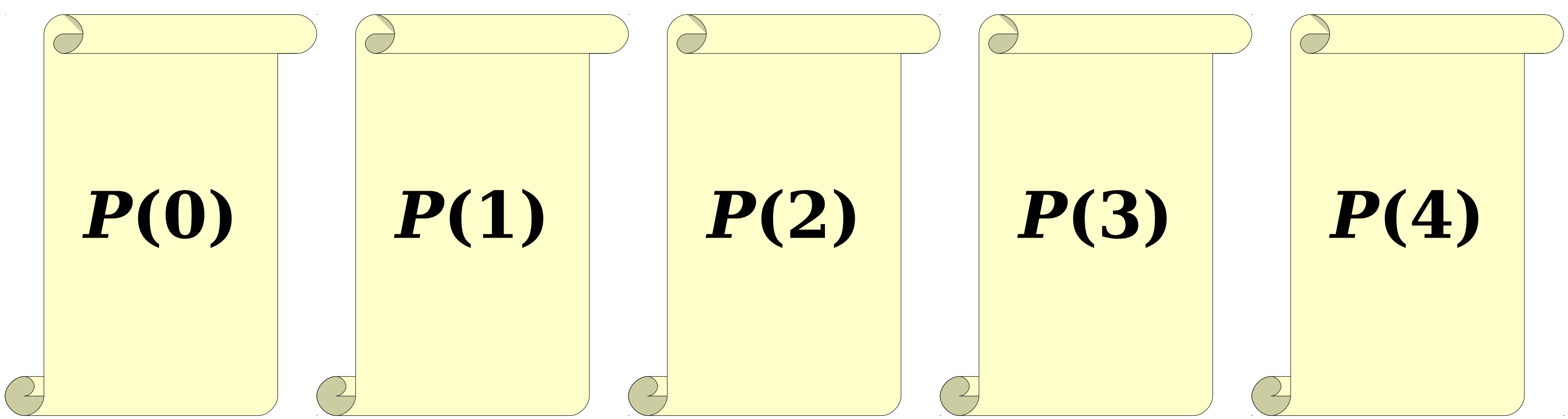

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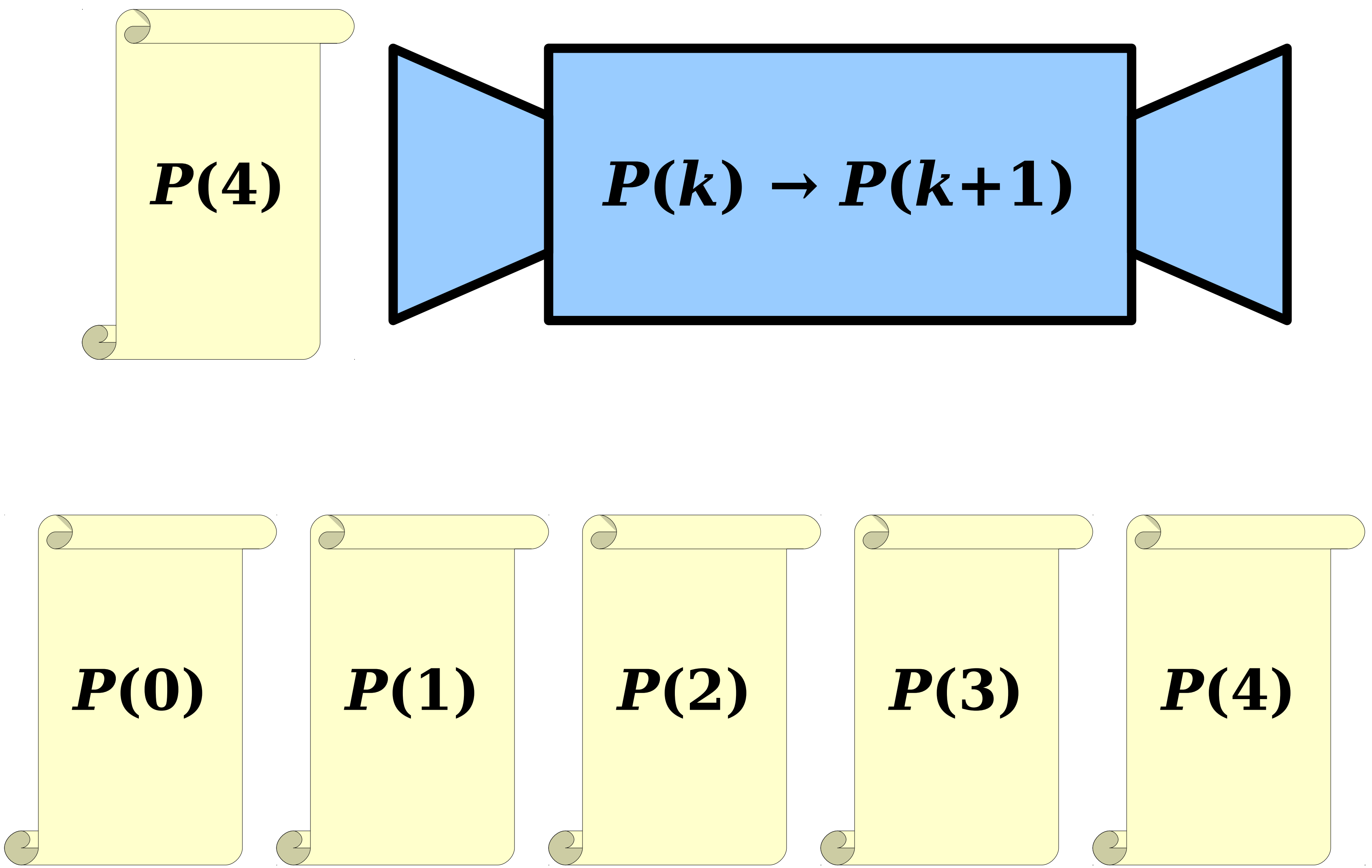

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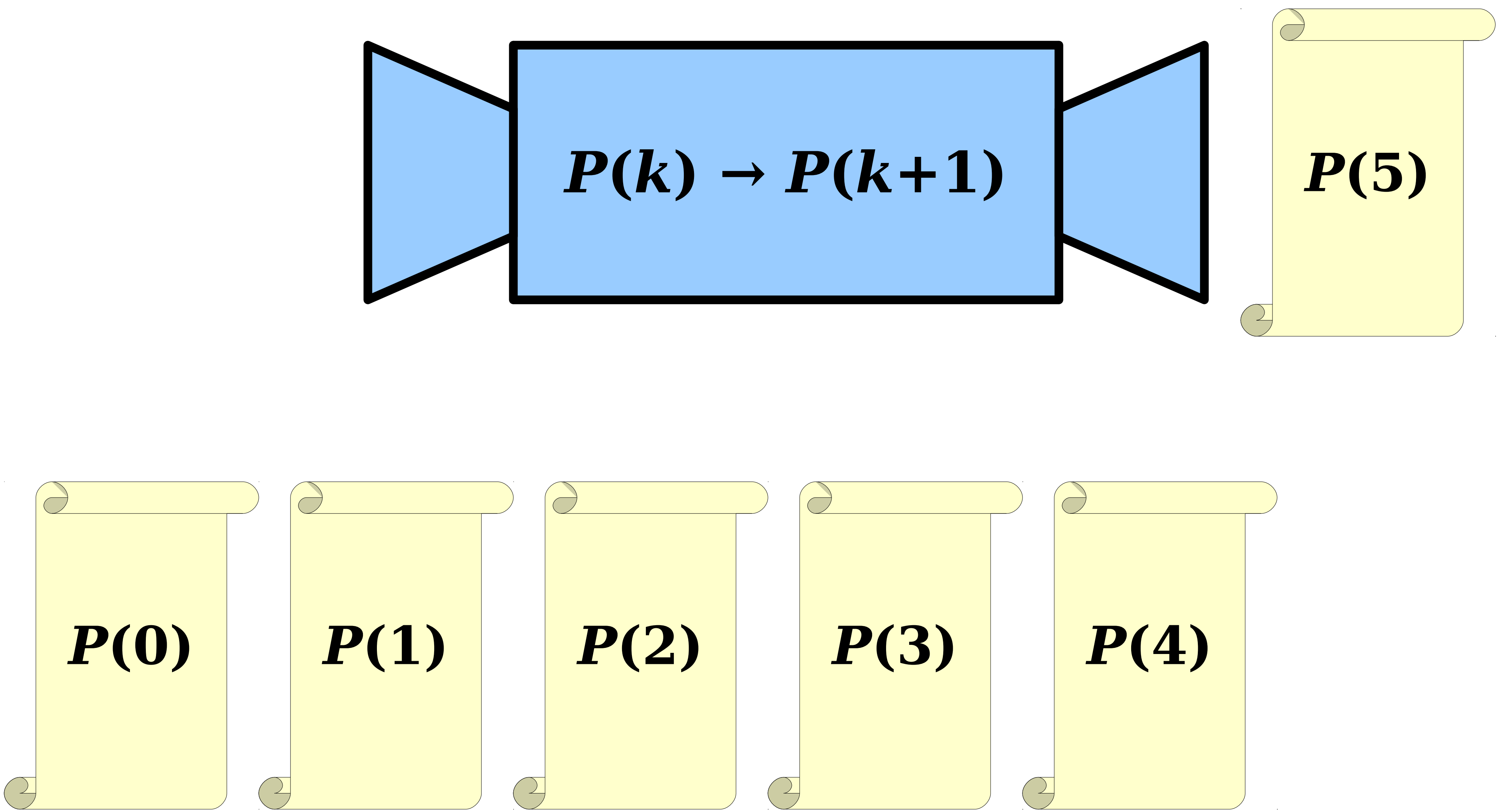

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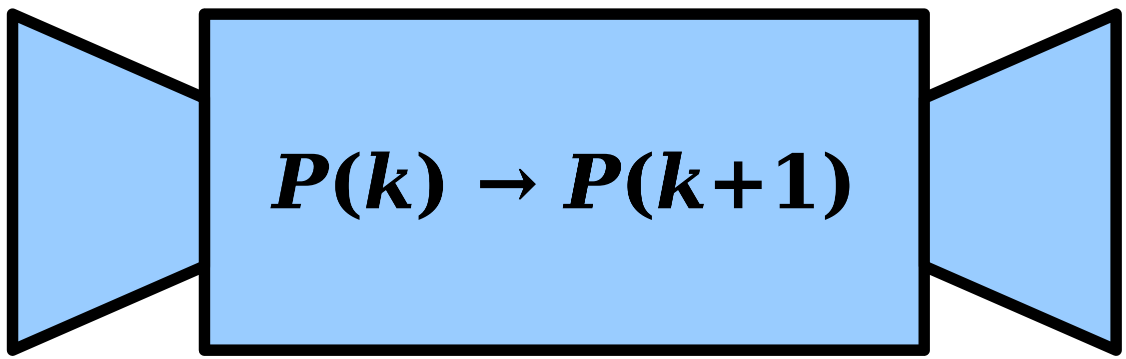
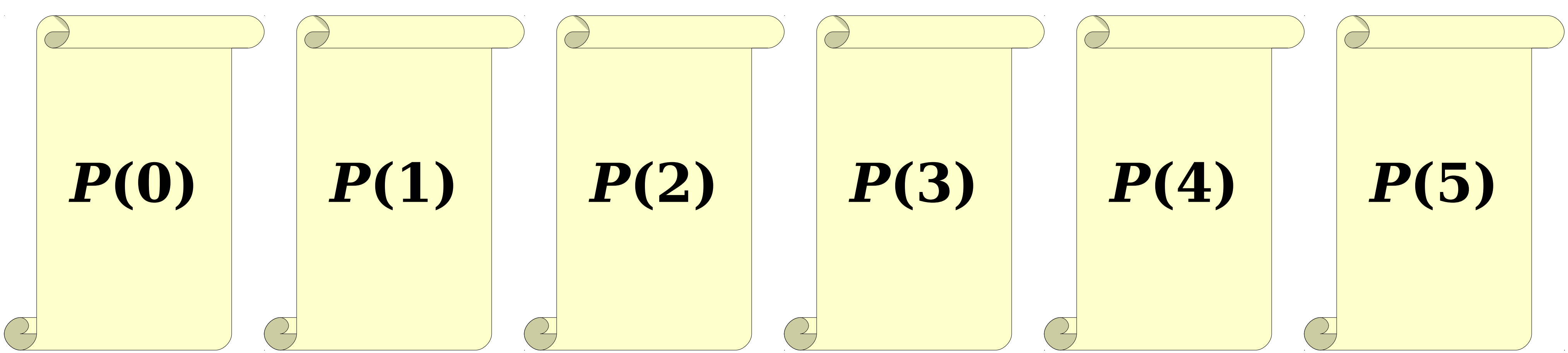
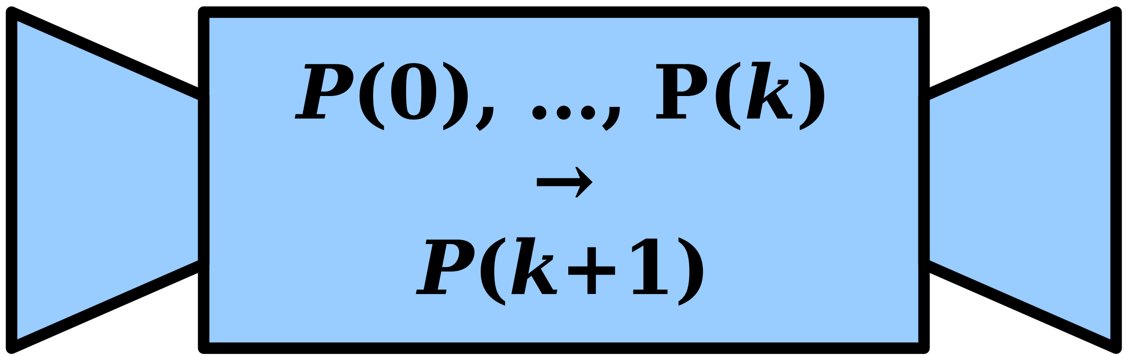
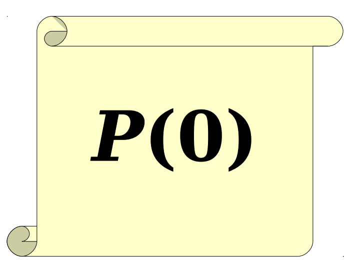




Intuiting Complete Induction





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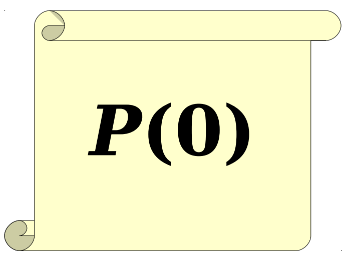

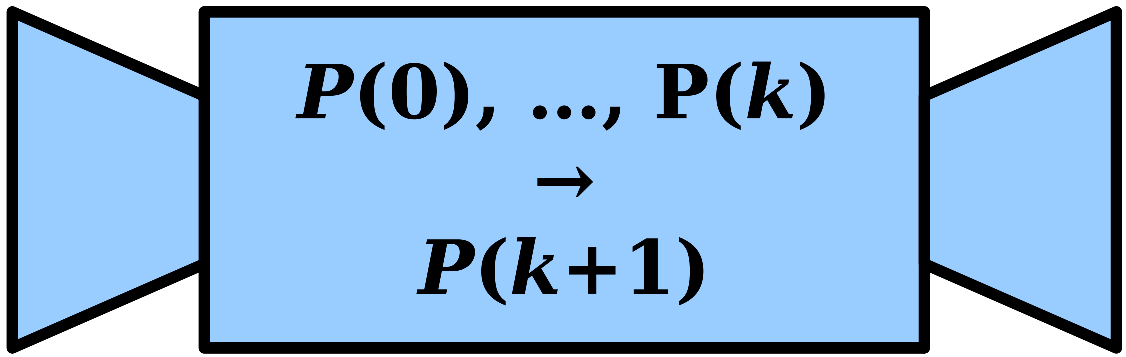

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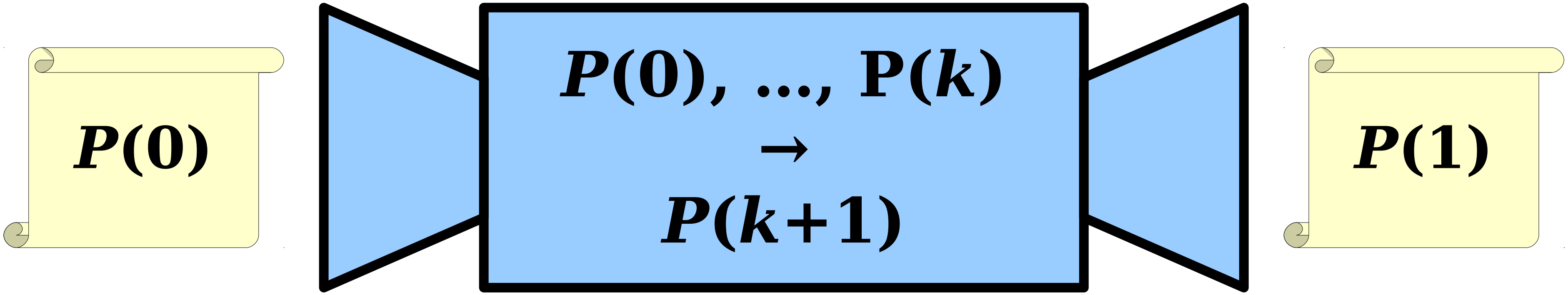

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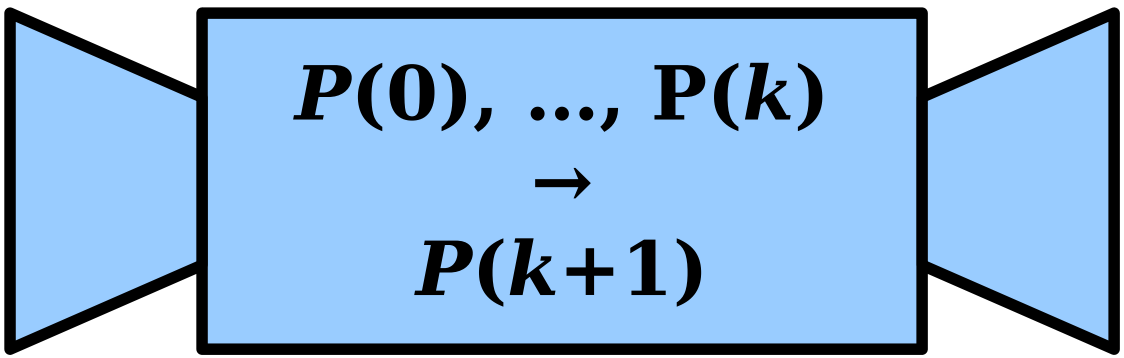

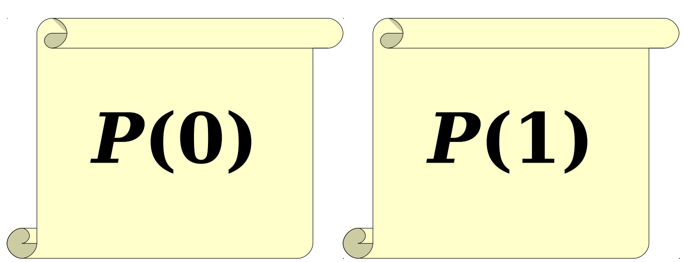

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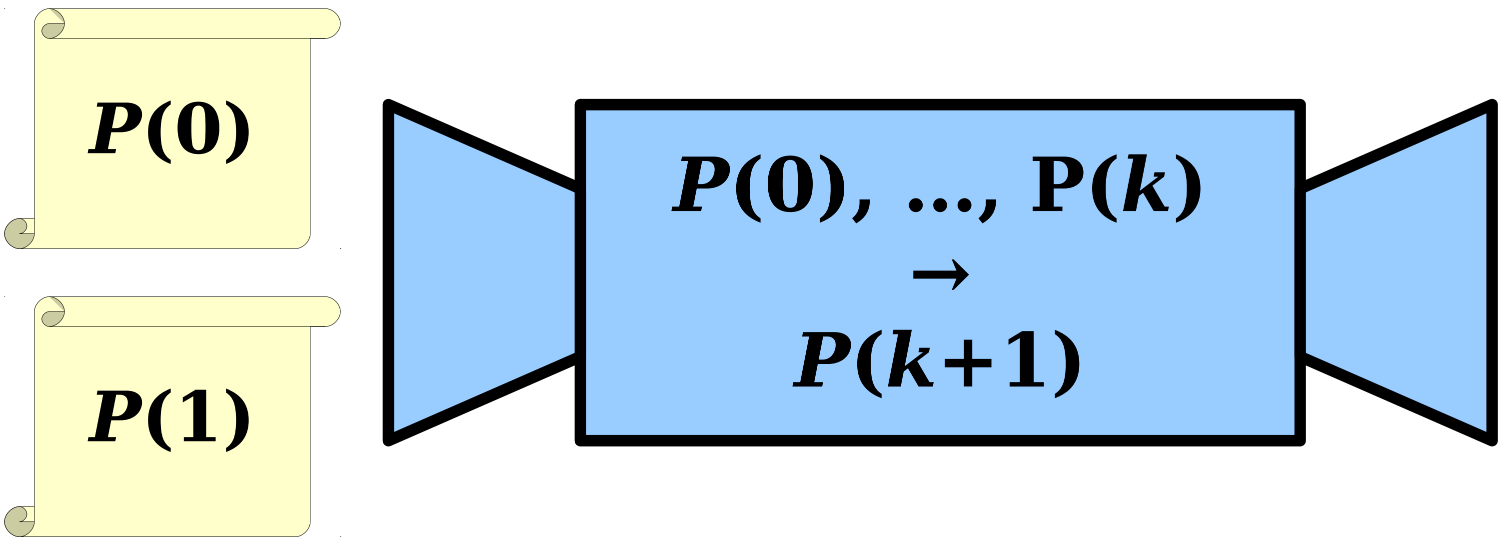

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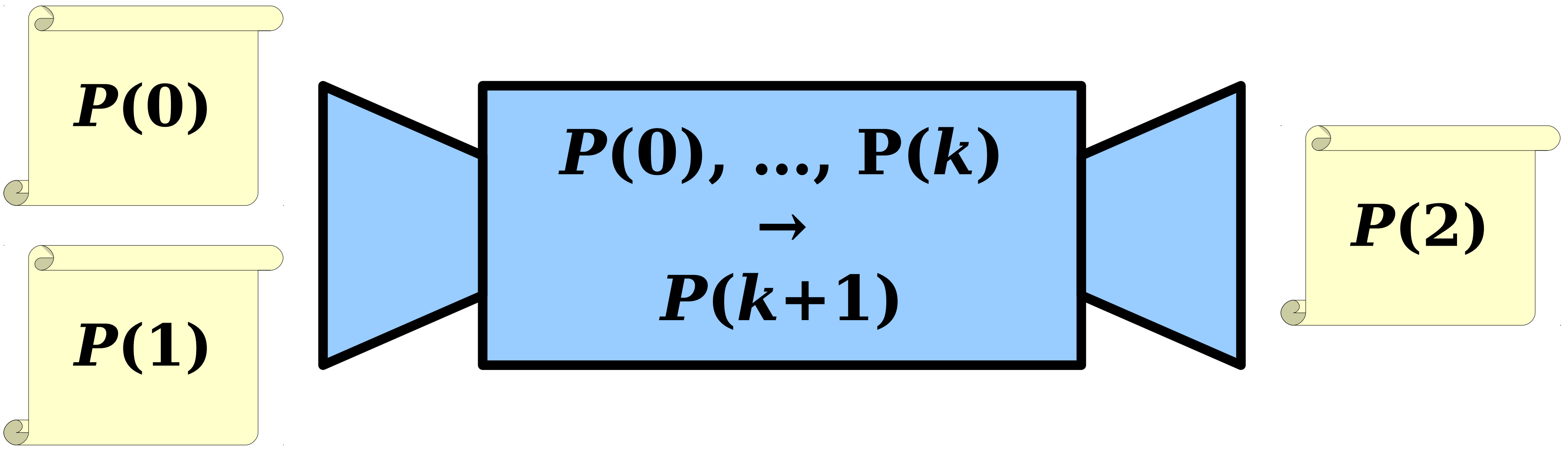

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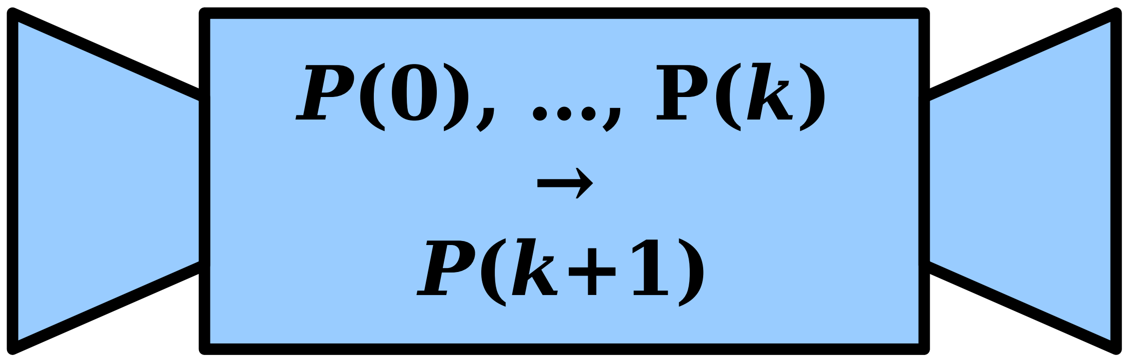

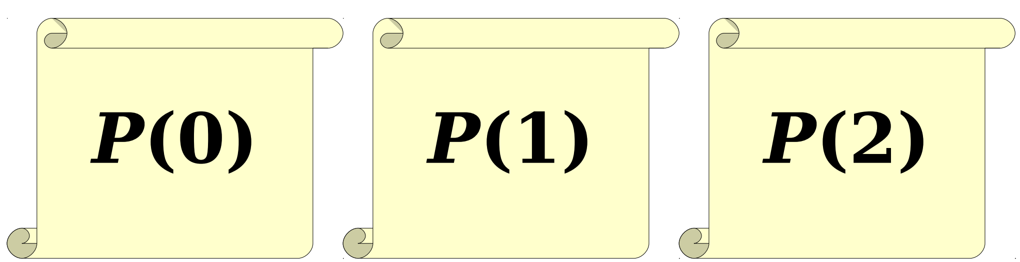

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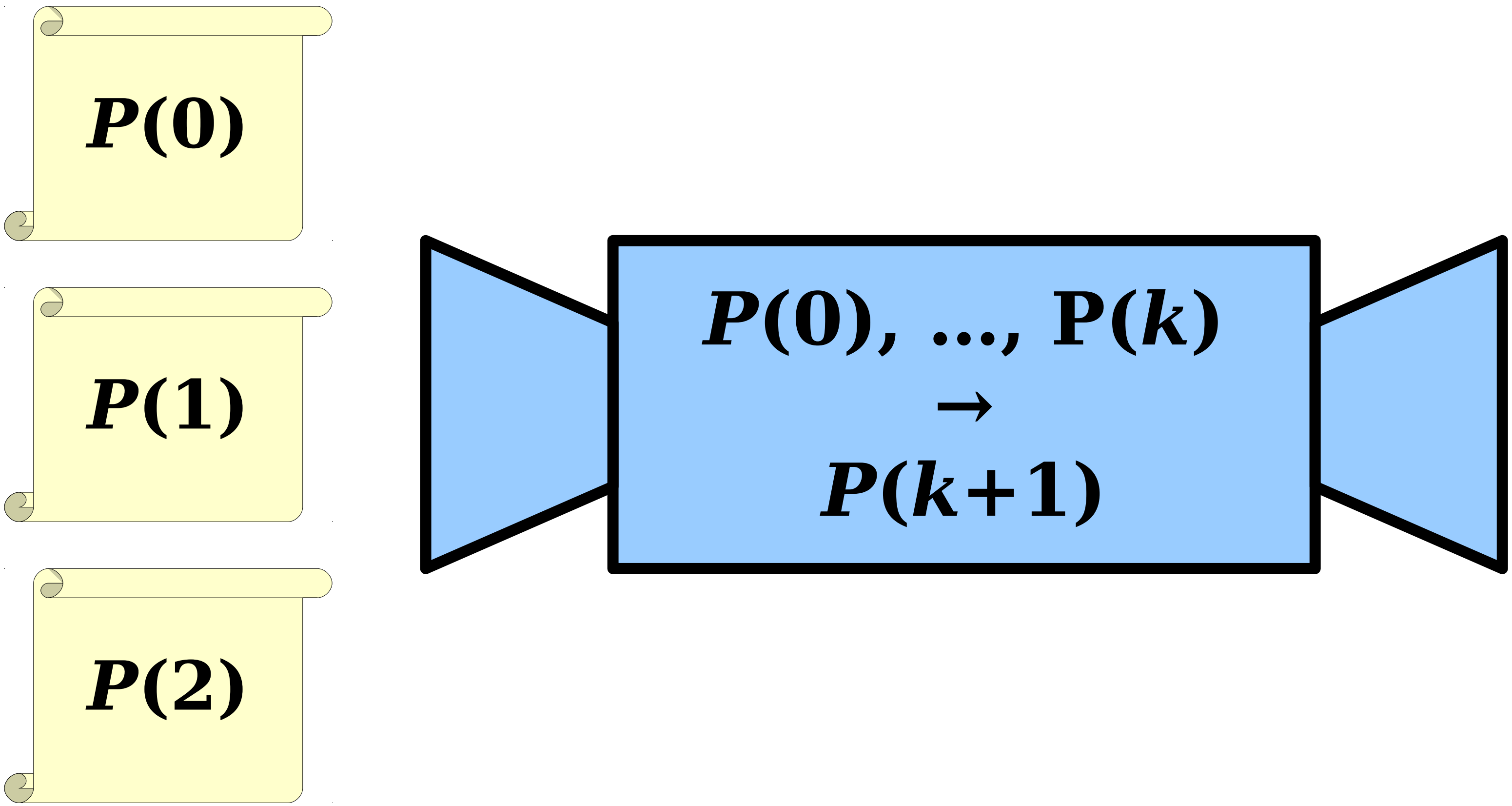

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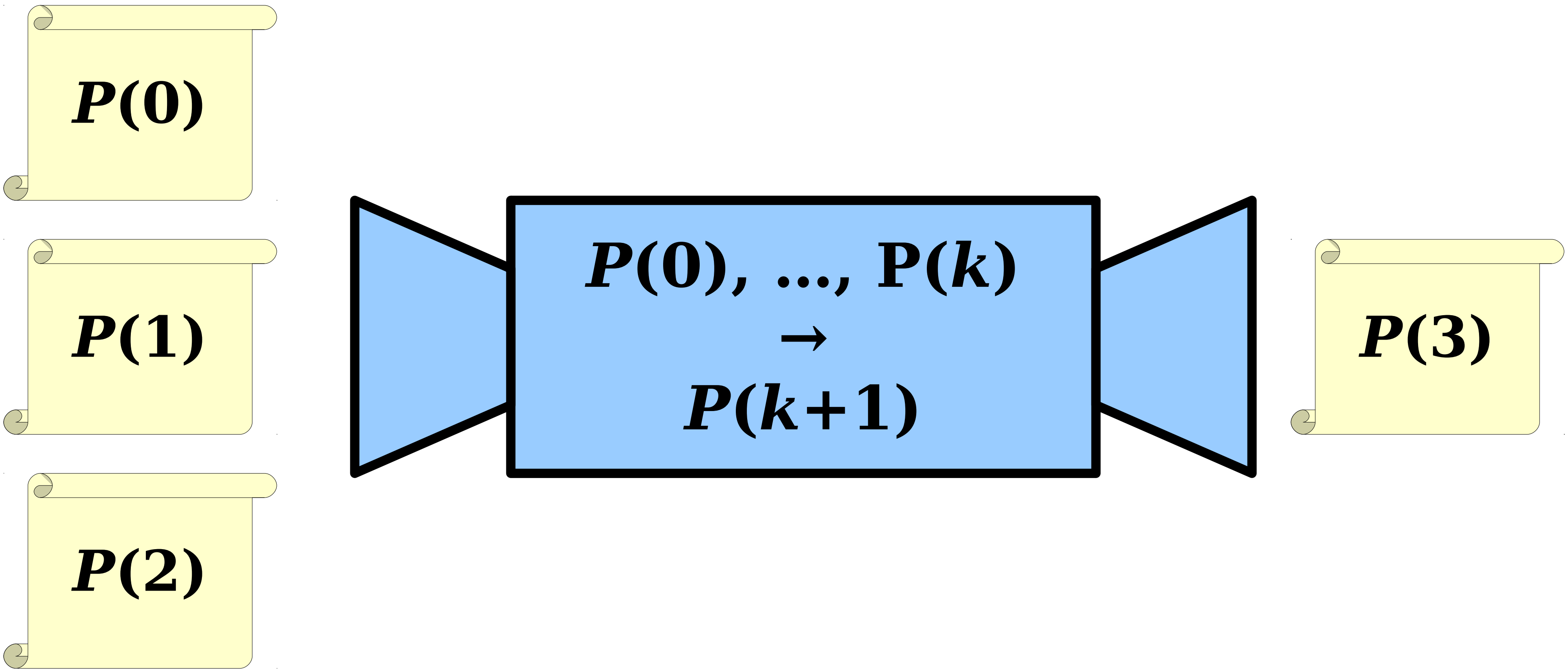

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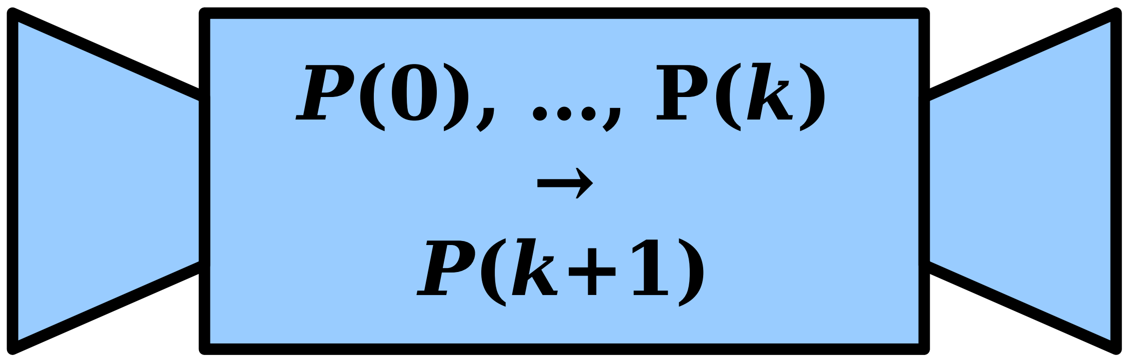

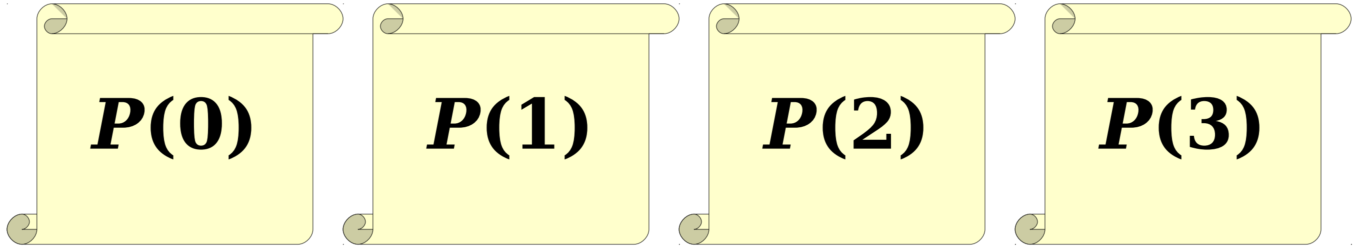

When Use Complete Induction?

- Normal induction is good for when you are shrinking the problem size by exactly one.
  - Peeling one final term off a sum.
  - Making one weighing on a scale.
  - Considering one more action on a string.
- Complete induction is good when you are shrinking the problem, but you can't be sure by how much.
  - In the previous example, if we delete a random edge, we can't know in advance how big the resulting trees will be.

For more on trees, take CS161 / 261 / 267!

An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Relations	Mathematical Logic
Set Theory	Cardinality

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.